Chapter 1

Logical Thinking

Mathematicians are in the business of stating things precisely. When you read a mathematical statement, you should take every word seriously; good mathematical language conveys a clear, unambiguous message. In order to read and write mathematics, you must practice the art of logical thinking. The goal of this chapter is to help you communicate mathematically by understanding the basics of logic.

A word of warning: mathematical logic can be difficult—especially the first time you see it. This chapter begins with the study of formal, or symbolic, logic, and then applies this study to the language of mathematics. Expect things to be a bit foggy at first, but eventually (we hope) the fog will clear. When it does, mathematical statements will start making more sense to you.

1.1 Formal Logic

Notation is an important part of mathematical language. Mathematicians' blackboards are often filled with an assortment of strange characters and symbols; such a display can be intimidating to the novice, but there's a good reason for communicating this way. Often, the act of reducing a problem to symbolic language helps us see what

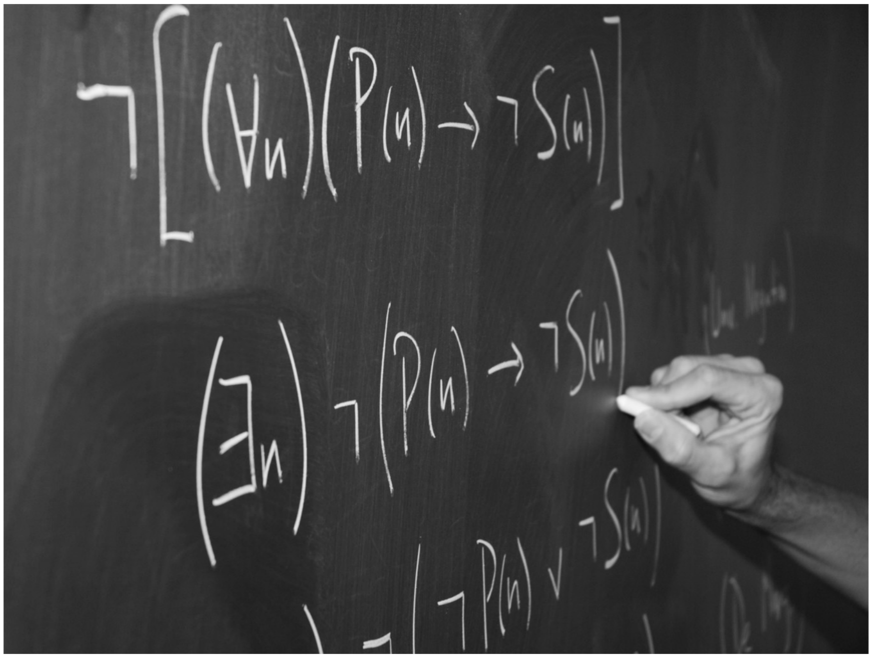


Figure 1.1 Symbols are an important part of the language of mathematics.

is really going on. Instead of operating in the fuzzy world of prose, we translate a problem to notation and then perform well-defined symbolic manipulations on that notation. This is the essence of the powerful tool called *formalism*. In this section, we explore how a formal approach to logic can help us avoid errors in reasoning.

A note on terminology: we'll use the word *formal* to describe a process that relies on manipulating notation. Often, people use this word to mean "rigorous," but that's not our intention. A formal argument can be rigorous, but so can an argument that does not rely on symbols.

One nice feature of formalism is that it allows you to work without having to think about what all the symbols mean. In this sense, formal logic is really "logical *not*-thinking." Why is this an advantage? Formal calculations are less prone to error. You are already familiar with this phenomenon: much of the arithmetic you learned in school was formal. You have a well-defined symbolic algorithm for multiplying numbers using pencil and paper, and you can quite effectively multiply three-digit numbers without thinking much about what you are really doing. Of course, formalism is pointless if you don't know what you are doing; at the end of any formal calculation, it is important to be able to interpret the results.

1.1.1 Preview Questions

Every section of this book begins with some open-ended *preview questions* that are designed to help you begin thinking about the ideas that follow. Often, these questions will introduce unfamiliar concepts, and they do not call for complete solutions. To get the most out of these problems, spend a little time thinking about them on your own, write down some preliminary responses, and make a note of any questions that you have.

Preview 1.1

1. Westley, standing with his hands behind his back, claims that he is holding a quarter in his left hand and a $20 bill in his right hand. You believe he is lying. What would you have to show to demonstrate that he is lying? Invent a diagram, chart, or symbols to illustrate all the possible scenarios.
2. Buttercup knows whether or not Westley is lying. She promises that if Westley is lying, she will give you a cookie. Buttercup always keeps her promises. Suppose she does not give you a cookie; what can you conclude? Suppose she gives you a cookie; what can you conclude? Illustrate your thinking in some organized way.
3. Camp Halcyon and Camp Placid are two summer camps with the following daily policies on pool use and cleanup duties.

   > *Camp Halcyon's Policy:* If you used the pool in the afternoon and you didn't clean up after lunch, then you must clean up after dinner.
   >
   > *Camp Placid's Policy:* You must do at least one of the following: (a) Stay out of the pool in the afternoon, (b) clean up after lunch, or (c) clean up after dinner.

   How do these policies differ?

1.1.2 Connectives and Propositions

In order to formalize logic, we need a system for translating statements into symbols. We'll start with a precise definition of *statement*.

Definition 1.1 A *statement* (also known as a *proposition*) is a declarative sentence that is either true or false, but not both.

The following are examples of statements:

- 7 is odd.
- $1 + 1 = 4$
- If it is raining, then the ground is wet.
- Our professor is from Mars.

Note that we don't need to be able to decide whether a statement is true or false in order for it to be a statement. Either our professor is from Mars or our professor is not from Mars, though we may not be sure which is the case.

How can a declarative sentence fail to be a statement? There are two main ways. A declarative sentence may contain an unspecified term:

x is even.

In this case, x is called a *free variable*. The truth of the sentence depends on the value of x, so if that value is not specified, we can't regard this sentence as a statement. A second type of declarative non-statement can happen when a sentence is *self-referential*:

This sentence is false.

We can't decide whether or not the above sentence is true. If we say it is true, then it claims to be false; if we say it is false, then it appears to be true.

Often, a complicated statement consists of several simple statements joined together by words such as "and," "or," "if . . . then," etc. These connecting words are represented by the five *logical connectives* shown in the following table. Logical connectives are useful for decomposing compound statements into simpler ones, because they highlight important logical properties of a statement.

Name	Symbol
and	$\wedge$
or	$\vee$
not	$\neg$
implies (if . . . then)	$\to$
if and only if	$\leftrightarrow$

In order to use a formal system for logic, we must be able to *translate* between a statement in English and its formal counterpart. We do this by assigning letters for simple statements and then building expressions with connectives.

Example 1.1 If p is the statement "you are wearing shoes" and q is the statement "you can't cut your toenails," then $p \to q$ represents the statement, "If you are wearing shoes, then you can't cut your toenails." We may choose to express this statement differently in English: "You can't cut your toenails if you are wearing shoes," or "Wearing shoes makes it impossible to cut your toenails." The statement $\neg q$ translates literally as "It is not the case that you can't cut your toenails." Of course, in English, we would prefer to say simply, "You can cut your toenails," but this involves using logic, as we will see in the next section.

1.1.3 Truth Tables

We haven't finished setting up our formal system for logic because we haven't been specific about the meaning of the logical connectives. Of course, the names of each connective suggest how they should be used, but in order to make statements with mathematical precision, we need to know exactly what each connective means.

Defining the meaning of a mathematical symbol is harder than it might seem. Even the $+$ symbol from ordinary arithmetic is problematic. Although we all have an intuitive understanding of addition—it describes how to combine two quantities—it is hard to express this concept in words without appealing to our intuition. What does "combine" mean, exactly? What are "quantities," really?

One simple, but obviously impractical, way to define the $+$ sign would be to list all possible addition problems, as in the following table. Of course, such a table could never end, but it would, in theory, give us a precise definition of the $+$ sign.

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	2
2	1	3
$\vdots$	$\vdots$	$\vdots$

The situation in logic is easier to handle. Any statement has two possible values: true (T) or false (F). So when we use variables such as p or q for statements in logic, we can think of them as unknowns that can take one of only two values: T or F. This makes it possible to define the meaning of each connective using tables; instead of infinitely many values for numbers x and y, we have only two choices for each variable p and q.

We will now stipulate the meaning of each logical connective by listing the T/F values for every possible case. The simplest example is the "not" connective, $\neg$. If p is true, then $\neg p$ should be false, and *vice versa*.

p	$\neg p$
T	F
F	T

This table of values is called a *truth table*; it defines the T/F values for the connective.

The "and" and "or" connectives are defined by the following truth tables. Since we have two variables, and each can be either T or F, we need four cases.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The definition of the "and" connective $\wedge$ is what you would expect: in order for $p \wedge q$ to be true, p must be true *and* q must be true. The "or" connective $\vee$ is a little less obvious. Notice that our definition stipulates that $p \vee q$ is true whenever p is true, or q is true, or both are true. This can be different from the way "or" is used in everyday speech. When you are offered "soup or salad" in a restaurant, your server isn't expecting you to say "both."

The "if and only if" connective says that two statements have exactly the same truth values. Thus, its truth table is as follows.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Sometimes authors will write "iff" as an abbreviation for "if and only if."

The "if . . . then" connective has the least intuitive definition.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

To understand the motivation for this definition, let $p \to q$ be the statement of Example 1.1:

> "If you are wearing shoes, then you can't cut your toenails."

In order to demonstrate that this statement is false, you would have to be able to cut your toenails while wearing shoes. In any other situation, you would have to concede that the statement is not false (and if a statement is not false, it must be true). If you are not wearing shoes, then maybe you can cut your toenails or maybe you can't, for some other reason. This doesn't contradict the statement $p \to q$.

Put another way, if you live in a world without shoes, then the statement is *vacuously* true; since you can never actually wear shoes, it isn't false to say that "*If* you are wearing shoes," then anything is possible. This explains the last two lines of the truth table; if p is false, then $p \to q$ is true, no matter what q is.

Often, mathematicians use the word "implies" as a synonym for the $\to$ connective. "If p then q" means the same thing as "p implies q," namely that q is a necessary consequence of p. Like many words in the English language, "imply" has multiple meanings. Sometimes it means "to indicate or suggest," as in, "She didn't say she wanted to leave, but she implied it." The mathematical usage is stronger, expressing a forced relationship: "$x > 3$ implies $x^2 > 3$." It is important to recognize when common words have special meanings in mathematical writing; Exercise 32 at the end of this section explores another example, the word "only."

1.1.4 Activities

The *activity* below will help you practice using truth tables while previewing some of the results that we will discuss later. In the same way, the activities that you will encounter throughout this text will encourage you to process new ideas and explore their consequences. Ideally, you will work cooperatively with others on these activities, so you can share ideas and compare answers.

Activity 1.1.1: Truth Tables

1. Complete the missing columns in the following truth table.

p	q	$p \vee q$	$p \wedge q$	$p \to q$	$\neg p$	$\neg p \vee q$	$\neg q$	$\neg q \to \neg p$	$\neg(p \wedge q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg p \vee \neg q$
T	T	T	T	T	F							
T	F	T	F	F	F							
F	T	T	F	T	T							
F	F	F	F	T	T							

2. In the truth table above, find three different statements that have the same T/F values for all cases.
3. Compare the truth tables for $\neg(p \wedge q)$, $\neg(p \vee q)$, $\neg p \wedge \neg q$, and $\neg p \vee \neg q$. Is there a rule for logical connectives that resembles the distributive property from algebra?

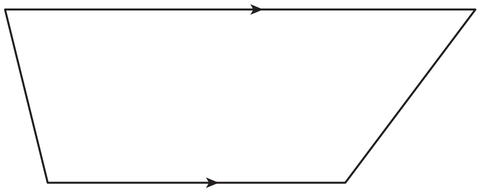
1.1.5 Logical Equivalences

Definition 1.2 Two statements are *logically equivalent* if they have the same T/F values for all cases, that is, if they have the same truth tables.

There are some logical equivalences that come up often in mathematics, and also in life in general.

Example 1.2 Consider the following theorem from high school geometry. (Recall that two angles are *supplementary* if their angle measures sum to 180°.)

> If a quadrilateral has a pair of parallel sides, then it has a pair of supplementary angles.



This theorem is of the form $p \to q$, where p is the statement that the quadrilateral has a pair of parallel sides, and q is the statement that the quadrilateral has a pair of supplementary angles.

We can state a different theorem, represented by $\neg q \to \neg p$.

> If a quadrilateral does not have a pair of supplementary angles, then it does not have a pair of parallel sides.

We know that this second theorem is logically equivalent to the first because the formal statement $p \to q$ is logically equivalent to the formal statement $\neg q \to \neg p$, as the following truth table shows.

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Notice that the column for $p \to q$ matches the column for $\neg q \to \neg p$. Since the first theorem is a true theorem from geometry, so is the second.

Now consider a different variation on this theorem.

> If a quadrilateral has a pair of supplementary angles, then it has a pair of parallel sides.

It turns out that this statement is not true, in general, in geometry. (Can you draw an example of a quadrilateral for which it fails to be true?) In propositional logic, this statement is of the form $q \to p$. The following truth table shows that $q \to p$ is *not* logically equivalent to $p \to q$, because the T/F values are different in the second and third rows.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

The statement $\neg q \to \neg p$ is called the *contrapositive* of $p \to q$, and the statement $q \to p$ is called the *converse*. The truth tables above prove that, for any statement s, the contrapositive of s is logically equivalent to s, while the converse of s may not be.

There are lots of situations where assuming the converse can cause trouble. For example, suppose that the following statement is true.

> If a company is not participating in illegal accounting practices, then an audit will turn up no evidence of wrongdoing.

It is certainly reasonable to assume this, since there couldn't be evidence of wrongdoing if no such wrongdoing exists. However, the converse is probably not true:

> If an audit turns up no evidence of wrongdoing, then the company is not participating in illegal accounting practices.

After all, it is possible that the auditors missed something.

At this point, you might object that formal logic seems like a lot of trouble to go through just to verify deductions like this last example. This sort of thing is just common sense, right? Well, maybe. But something that appears obvious to you may not be obvious to someone else. Furthermore, our system of formal logic can deal with more complicated situations, where our common sense might fail us. The solution to the next example uses formal logic. Before you look at this solution, try to solve the problem using "common sense." Although the formal approach takes a little time, it resolves any doubt you might have about your own reasoning process.

Example 1.3 If Aaron is late, then Bill is late, and, if both Aaron and Bill are late, then class is boring. Suppose that class is not boring. What can you conclude about Aaron?

Solution: Let's begin by translating the first sentence into the symbols of logic, using the following statements.

$$p = \text{“Aaron is late.”}$$
$$q = \text{“Bill is late.”}$$
$$r = \text{“Class is boring.”}$$

Let S be the statement "If Aaron is late, then Bill is late, and, if both Aaron and Bill are late, then class is boring." In symbols, S translates to the following.

$$S = (p \to q) \wedge [(p \wedge q) \to r]$$

Now let's construct a truth table for S. We do this by constructing truth tables for the different parts of S, starting inside the parentheses and working our way out.

Row #	p	q	r	$p \to q$	$p \wedge q$	$(p \wedge q) \to r$	S
1.	T	T	T	T	T	T	T
2.	T	T	F	T	T	F	F
3.	T	F	T	F	F	T	F
4.	T	F	F	F	F	T	F
5.	F	T	T	T	F	T	T
6.	F	T	F	T	F	T	T
7.	F	F	T	T	F	T	T
8.	F	F	F	T	F	T	T

You should check that the last column is the result of "and-ing" the column for $p \to q$ with the column for $(p \wedge q) \to r$.

We are interested in the possible values of p. It is given that S is true, so we can eliminate rows 2, 3, and 4, the rows where S is false. If we further assume that class is not boring, we can also eliminate the rows where r is true, namely the odd-numbered rows. The rows that remain are the only possible T/F values for p, q, and r: rows 6 and 8. In both of these rows, p is false. In other words, Aaron is not late. ◇

Activity 1.1.2: Truth Tables and Deduction

Let the following statements be given.

p = "You used the pool in the afternoon."

q = "You cleaned up after lunch."

r = "You must clean up after dinner."

1. Use connectives to translate the following statement into formal logic.

   If you used the pool in the afternoon and you didn't clean up after lunch, then you must clean up after dinner.

2. Construct a truth table for the formal logic statement that you found in part (1). The eight rows of your table should correspond to the eight different possibilities for p, q, and r.
3. Suppose that the statement given in part (1) is false. What must be true about your pool usage and cleanup duties? Explain how to justify your answer using the truth table.

Exercises 1.1

1. Let the following statements be given.

   p = "There is water in the cylinders."

   q = "The head gasket is blown."

   r = "The car will start."

   (a) Translate the following statement into symbols of formal logic.

   If the head gasket is blown and there's water in the cylinders, then the car won't start.

   (b) Translate the formal statement $r \to \neg(q \vee p)$ into everyday English.

2. Let the following statements be given.

$$p = \text{“You are in Seoul.”}$$
$$q = \text{“You are in Kwangju.”}$$
$$r = \text{“You are in South Korea.”}$$

   (a) Translate the following statement into symbols of formal logic.

   If you are not in South Korea, then you are not in Seoul or Kwangju.

   (b) Translate the formal statement $q \to (r \wedge \neg p)$ into everyday English.

3. Let the following statements be given.

$$p = \text{“You can vote.”}$$
$$q = \text{“You are under 18 years old.”}$$
$$r = \text{“You are from Mars.”}$$

   (a) Translate the following statement into symbols of formal logic.

   You can't vote if you are under 18 years old or you are from Mars.

   (b) Give the contrapositive of this statement in the symbols of formal logic.
   (c) Give the contrapositive in English.

4. Let s be the following statement.

   If you are studying hard, then you are staying up late at night.

   (a) Give the converse of s.
   (b) Give the contrapositive of s.

5. Let s be the following statement.

   If it is raining, then the ground is wet.

   (a) Give the converse of s.
   (b) Give the contrapositive of s.

6. Give an example of a quadrilateral that shows that the *converse* of the following statement is false.

   If a quadrilateral has a pair of parallel sides, then it has a pair of supplementary angles.

7. We say that two ordered pairs (a, b) and (c, d) are *equal* when $a = c$ and $b = d$. Let s be the following statement.

   If $(a, b) = (c, d)$, then $a = c$.

   (a) Is this statement true?
   (b) Write down the converse of s.
   (c) Is the converse of s true? Explain.

8. Give an example of a true if–then statement whose converse is also true.
9. Show that $p \leftrightarrow q$ is logically equivalent to $(p \to q) \wedge (q \to p)$ using truth tables.
10. Use truth tables to establish the following equivalences.

    (a) Show that $\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$.
    (b) Show that $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$.

    These equivalences are known as *De Morgan's laws*, after the 19^{th}-century logician Augustus De Morgan.

11. Are the statements $\neg(p \to q)$ and $\neg p \to \neg q$ logically equivalent? Justify your answer using truth tables.
12. Use truth tables to show that $(a \vee b) \wedge (\neg(a \wedge b))$ is logically equivalent to $a \leftrightarrow \neg b$. (This arrangement of T/F values is sometimes called the *exclusive or* of a and b.)

13. Use a truth table to prove that the statement $[(p \vee q) \wedge (\neg p)] \rightarrow q$ is always true, no matter what p and q are.
14. Let the following statements be given.

$$p = \text{"Andy is hungry."}$$
$$q = \text{"The refrigerator is empty."}$$
$$r = \text{"Andy is mad."}$$

   (a) Use connectives to translate the following statement into formal logic.

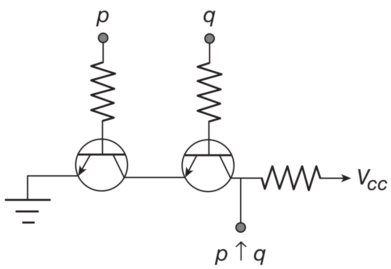
   If Andy is hungry and the refrigerator is empty, then Andy is mad.

   (b) Construct a truth table for the statement in part (a).
   (c) Suppose that the statement given in part (a) is true, and suppose also that Andy is not mad and the refrigerator is empty. Is Andy hungry? Explain how to justify your answer using the truth table.
15. Let A be the statement $p \rightarrow (q \wedge \neg r)$. Let B be the statement $q \leftrightarrow r$.
   (a) Construct truth tables for A and B.
   (b) Suppose statements A and B are both true. What can you conclude about statement p? Explain your answer using the truth table.
16. Use truth tables to prove the following *distributive properties* for propositional logic.
   (a) $p \wedge (q \vee r)$ is logically equivalent to $(p \wedge q) \vee (p \wedge r)$.
   (b) $p \vee (q \wedge r)$ is logically equivalent to $(p \vee q) \wedge (p \vee r)$.
17. Use truth tables to prove the *associative properties* for propositional logic.
   (a) $p \vee (q \vee r)$ is logically equivalent to $(p \vee q) \vee r$.
   (b) $p \wedge (q \wedge r)$ is logically equivalent to $(p \wedge q) \wedge r$.
18. Mathematicians say that "statement P is *stronger* than statement Q" if Q is true whenever P is true, but not conversely. (In other words, "P is stronger than Q" means that $P \rightarrow Q$ is always true, but $Q \rightarrow P$ is not true, in general.) Use truth tables to show the following.
   (a) $a \wedge b$ is stronger than a.
   (b) a is stronger than $a \vee b$.
   (c) $a \wedge b$ is stronger than $a \vee b$.
   (d) b is stronger than $a \rightarrow b$.
19. Suppose Q is a quadrilateral. Which statement is stronger? Explain.
   - Q is a square.
   - Q is a rectangle.
20. Which statement is stronger? Explain.
   - Manchester United is the best football team in England.
   - Manchester United is the best football team in Europe.
21. Which statement is stronger? Explain.
   - n is divisible by 3.
   - n is divisible by 12.
22. Mathematicians say that "Statement P is a *sufficient condition* for statement Q" if $P \rightarrow Q$ is true. In other words, in order to know that Q is true, it is sufficient to know that P is true. Let x be an integer. Give a sufficient condition on x for $x/2$ to be an even integer.
23. Mathematicians say that "Statement P is a *necessary condition* for statement Q" if $Q \rightarrow P$ is true. In other words, in order for Q to be true, P must be true. Let $n \geq 1$ be a natural number. Give a necessary but not sufficient condition on n for $n + 2$ to be prime.

24. Let Q be a quadrilateral. Give a sufficient but not necessary condition for Q to be a parallelogram.
25. Write the statement "P is necessary and sufficient for Q" in the symbols of formal logic, using as few connectives as possible.
26. Often a complicated expression in formal logic can be simplified. For example, consider the statement $S = (p \wedge q) \vee (p \wedge \neg q)$.
    (a) Construct a truth table for S.
    (b) Find a simpler expression that is logically equivalent to S.
27. Consider the statement $S = [\neg(p \to q)] \vee [\neg(p \vee q)]$.
    (a) Construct a truth table for S.
    (b) Find a simpler expression that is logically equivalent to S.
28. The NAND connective $\uparrow$ is defined by the following truth table. Use truth tables to show that $p \uparrow q$ is logically equivalent to $\neg(p \wedge q)$. (This explains the name NAND: Not AND.)

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

29. The NAND connective is important because it only takes two transistors to build an electronic circuit that computes the NAND of two signals:



Such a circuit is called a *logic gate*. Moreover, it is possible to build logic gates for the other logical connectives entirely out of NAND gates. Prove this fact by proving the following equivalences, using truth tables.
    (a) $(p \uparrow q) \uparrow (p \uparrow q)$ is logically equivalent to $p \wedge q$.
    (b) $(p \uparrow p) \uparrow (q \uparrow q)$ is logically equivalent to $p \vee q$.
    (c) $p \uparrow (q \uparrow q)$ is logically equivalent to $p \to q$.
30. Write $\neg p$ in terms of p and $\uparrow$.
31. A technician suspects that one or more of the processors in a distributed system is not working properly. The processors, A, B, and C, are all capable of reporting information about the status (working or not working) of the processors in the system. The technician is unsure whether a processor is really not working, or whether the problem is in the status reporting routines in one or more of the processors. After polling each processor, the technician receives the following status reports.

- Processor A reports that processor B is not working and processor C is working.
- Processor B reports that A is working if and only if B is working.
- Processor C reports that at least one of the other two processors is not working.

Help the technician by answering the following questions.
    (a) Let a = "A is working," b = "B is working," and c = "C is working." Write the three status reports in terms of a, b, and c, using the symbols of formal logic.
    (b) Complete the following truth table.

a	b	c	A's report	B's report	C's report
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

(c) Assuming that all of the status reports are true, which processor(s) is/are working?

(d) Assuming that all of the processors are working, which status report(s) is/are false?

(e) Assuming that a processor's status report is true if and only if the processor is working, what is the status of each processor?

32. Use the symbols of propositional logic to explain the difference between the following two statements.

    My team will win if I yell at the TV.
    My team will win only if I yell at the TV.

    Look up the word "only" in a dictionary. This word has several different meanings. Which meaning applies when we use the phrase "if and only if" in logic?

1.2 Propositional Logic

After working through the exercises of the previous section, you may have noticed a serious limitation of the truth table approach. Each time you add a new statement to a truth table, you must double the number of rows. This makes truth table analysis unwieldy for all but the simplest examples.

In this section, we will develop a system of rules for manipulating formulas in symbolic logic. This system, called the *propositional calculus*, will allow us to make logical deductions formally. There are at least three reasons for doing this.

1. These formal methods are useful for analyzing complex logical problems, especially where truth tables are impractical.
2. The derivation rules we will study are commonly used in mathematical discourse.
3. The system of derivation rules and proof sequences is a simple example of mathematical proof.

Of these three, the last is the most important. The mechanical process of writing proof sequences in propositional calculus will prepare us for writing more complicated proofs in other areas of mathematics.

1.2.1 Tautologies and Contradictions

Preview 1.2

1. Explain how the answers to the following two questions are related.

    If you pass all the exams, will you pass the course?

    Is it possible to pass all the exams and fail the course?

2. Consider the following statement.

    If you have a ticket, then, as long as you are wearing a shirt, you may enter the theater, unless you aren't wearing shoes.

    Write a simpler statement that expresses the same policy. Explain how you know that your statement is equivalent.

3. Suppose that a natural number n is *gaunt* if it satisfies the following condition.

   If n is even, then 10 divides n, and, if n is odd, then 5 divides n.

   List the first 6 gaunt numbers. Is there a simpler way to define the condition of "gauntness"?

There are some statements in formal logic that are always true, no matter what the T/F values of the component statements are. For example, the truth table for $(p \wedge q) \rightarrow p$ is as follows.

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Such a statement is called a *tautology*, and we write $(p \wedge q) \Rightarrow p$ to indicate this fact. The notation $A \Rightarrow B$ means that the statement $A \rightarrow B$ is true in all cases; in other words, the truth table for $A \rightarrow B$ is all Ts. Similarly, $A \Leftrightarrow B$ denotes that $A \leftrightarrow B$ is a tautology.

Example 1.4 In Activity 1.1.1 you proved the tautologies $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ and $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$.

When a tautology is of the form $(C \wedge D) \Rightarrow E$, we often prefer to write

$$\left.\begin{array}{r} C \\ D \end{array}\right\} \Rightarrow E$$

instead. This notation highlights the fact that if you know both C and D, then you can conclude E. The use of the $\wedge$ connective is implicit.

Example 1.5 Use a truth table to prove the following.

$$\left.\begin{array}{r} p \\ p \rightarrow q \end{array}\right\} \Rightarrow q$$

Solution: Let S be the statement $[p \wedge (p \rightarrow q)] \rightarrow q$. We construct our truth table by building up the parts of S, working from inside the parentheses outward.

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	S
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the column for S is all Ts, this proves that S is a tautology. ◇

The tautology in Example 1.5 is known as *modus ponens*, which is Latin for "affirmative mode." This concept goes back at least as far as the Stoic philosophers of ancient Greece, who stated it as follows.

> If the first, then the second;
> but the first;
> therefore the second.

In the exercises, you will have the opportunity to prove a related result called *modus tollens* ("denial mode"). In the symbols of logic, this tautology is as follows.

$$\left.\begin{array}{r} \neg q \\ p \rightarrow q \end{array}\right\} \Rightarrow \neg p$$

There are also statements in formal logic that are never true. A statement whose truth table contains all Fs is called a *contradiction*.

Example 1.6 Use a truth table to show that $p \wedge \neg p$ is a contradiction.

Solution:

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

In other words, a statement and its negation can never both be true. ◇

A statement in propositional logic that is neither a tautology nor a contradiction is called a *contingency*. A contingency has both Ts and Fs in its truth table, so its truth is "contingent" on the T/F values of its component statements. For example, $p \wedge q$, $p \vee q$, and $p \to q$ are all contingencies.

1.2.2 Derivation Rules

Tautologies are important because they show how one statement may be logically deduced from another. For example, suppose we know that the following statements are true.

> Our professor does not own a spaceship.
> If our professor is from Mars, then our professor owns a spaceship.

We can apply the *modus tollens* tautology to deduce that "Our professor is not from Mars." This is a valid argument, or *derivation*, that allows us to conclude this last statement given the first two.

Every tautology can be used as a rule to justify deriving a new statement from an old one. There are two types of derivation rules: equivalence rules and inference rules. Equivalence rules describe logical equivalences, while inference rules describe when a weaker statement can be deduced from a stronger statement. The equivalence rules given in Table 1.1 could all be checked using truth tables. If A and B are statements (possibly composed of many other statements joined by connectives), then the tautology $A \Leftrightarrow B$ is another way of saying that A and B are logically equivalent.[1]

Equivalence	Name
$p \Leftrightarrow \neg\neg p$	double negation
$p \to q \Leftrightarrow \neg p \vee q$	implication
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	associativity

Table 1.1 Equivalence Rules

An equivalence rule of the form $A \Leftrightarrow B$ can do three things:

1. Given A, deduce B.
2. Given B, deduce A.
3. Given a statement containing statement A, deduce the same statement, but with statement A replaced by statement B.

1. A word on notation: We typically use $p, q, r, \ldots$ to stand for simple statements, and we use $A, B, C, \ldots$ to denote statements that are (possibly) made up of simple statements and logical connectives. This convention, however, is purely expository and doesn't signify any difference in meaning.

The third option is a form of *substitution*. For example, given the following statement,

> If Micah is not sick and Micah is not tired, then Micah can play.

we can deduce the following using De Morgan's laws.

> If it is not the case that Micah is sick or tired, then Micah can play.

Activity 1.2.1: Using Equivalence Rules

In Activity 1.1.2, you translated the statement "If you used the pool in the afternoon and you didn't clean up after lunch, then you must clean up after dinner" into formal logic, using the following variables.

$$p = \text{"You used the pool in the afternoon."}$$
$$q = \text{"You cleaned up after lunch."}$$
$$r = \text{"You must clean up after dinner."}$$

1. Use the *implication* rule to rewrite your formal logic statement so that it doesn't contain the $\rightarrow$ connective.
2. Use *De Morgan's laws* to rewrite the resulting statement from part (1) so that it doesn't contain the $\wedge$ connective.
3. Use equivalence rules to simplify the *negation* of your statement from part (2) so that it doesn't use parentheses.
4. Without using truth tables, can you give an alternative explanation for your conclusion in part (3) of Activity 1.1.2?

In addition to equivalence rules, there are also inference rules for propositional logic. Unlike equivalence rules, inference rules work in only one direction. An inference rule of the form $A \Rightarrow B$ allows you to do only one thing:

1. Given A, deduce B.

In other words, you can conclude a weaker statement, B, if you have already established a stronger statement, A. For example, *modus tollens* is an inference rule: the weaker statement B:

> Our professor is not from Mars.

follows from the stronger statement A:

> Our professor does not own a spaceship, and if our professor is from Mars, then our professor owns a spaceship.

If A is true, then B must be true, but not *vice versa*. (Our professor might own a spaceship and be from Jupiter, for instance.) Table 1.2 lists some useful inference rules, all of which can be verified using truth tables.

Inference	Name
$\left.\begin{matrix} p \\ q \end{matrix}\right\} \Rightarrow p \wedge q$	conjunction
$\left.\begin{matrix} p \\ p \rightarrow q \end{matrix}\right\} \Rightarrow q$	*modus ponens*
$\left.\begin{matrix} \neg q \\ p \rightarrow q \end{matrix}\right\} \Rightarrow \neg p$	*modus tollens*
$p \wedge q \Rightarrow p$	simplification
$p \Rightarrow p \vee q$	addition

Table 1.2 Inference Rules

1.2.3 Proof Sequences

We now have enough tools to derive some new tautologies from old ones. A *proof sequence* is a sequence of statements and reasons to justify an assertion of the form $A \Rightarrow C$. The first statement, A, is given. (Often, there are several given statements.) The proof sequence can then list statements $B_1, B_2, B_3, \ldots$, etc., as long as each new statement can be derived from a previous statement (or statements) using some derivation rule. Of course, this sequence of statements must culminate in C, the statement we are trying to prove, given A.

Example 1.7 Write a proof sequence for the assertion

$$\left.\begin{array}{r} p \\ p \rightarrow q \\ q \rightarrow r \end{array}\right\} \Rightarrow r.$$

Solution:

Statements	Reasons
1. p	given
2. $p \rightarrow q$	given
3. $q \rightarrow r$	given
4. q	*modus ponens*, 1, 2
5. r	*modus ponens*, 4, 3

◇

Every time we prove something, we get a new inference rule. The rules in Table 1.2 are enough to get us started, but we should feel free to use proven assertions in future proofs. For example, the assertion proved in Example 1.7 illustrates the *transitive* property of the $\rightarrow$ connective.

Another thing to notice about Example 1.7 is that it was pretty easy—we just had to apply *modus ponens* twice. Compare this with the truth table approach: the truth table for

$$[p \wedge (p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow r$$

would consist of eight rows and several columns. Truth tables are easier to do, but they can be much more tedious.

Proof sequences should remind you of the types of proofs you did in high school geometry. The rules are simple: start with the given, see what you can deduce, and end with what you are trying to prove. Here's a harder example.

Example 1.8 Prove:

$$\left.\begin{array}{r} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

Solution:

Statements	Reasons
1. $p \vee q$	given
2. $\neg p$	given
3. $\neg(\neg p) \vee q$	double negation, 1
4. $\neg p \rightarrow q$	implication, 3
5. q	*modus ponens*, 4, 2

◇

Notice that in step 3 of this proof, we used one of the equivalence rules (double negation) to make a substitution in the formula. This is allowed: since $\neg(\neg p)$ is logically equivalent to p, it can take the place of p in any formula.

1.2.4 Forward–Backward

If you are having trouble coming up with a proof sequence, try the "forward–backward" approach: consider statements that are one step forward from the given, and also statements that are one step backward from the statement you are trying to prove. Repeat this process, forging a path of deductions forward from the given and backward from the final statement. If all goes well, you will discover a way to make these paths meet in the middle. The next example illustrates this technique.

Example 1.9 In Section 1.1, we used truth tables to show that a statement is logically equivalent to its contrapositive. In this example we will construct a proof sequence for one direction of this logical equivalence:

$$p \to q \;\Rightarrow\; \neg q \to \neg p$$

Solution: We apply the forward–backward approach. The only given statement is $p \to q$, so we search our derivation rules for something that follows from this statement. The only candidate is $\neg p \vee q$, by the implication rule, so we tentatively use this as the second step of the proof sequence. Now we consider the statement we are trying to prove, $\neg q \to \neg p$, and we look backward for a statement from which this statement follows. Since implication is an equivalence rule, we can also use it to move backward to the statement $\neg(\neg q) \vee \neg p$, which we propose as the second-to-last statement of our proof. By moving forward one step from the given and backward one step from the goal, we have reduced the task of proving

$$p \to q \;\Rightarrow\; \neg q \to \neg p$$

to the (hopefully) simpler task of proving

$$\neg p \vee q \;\Rightarrow\; \neg(\neg q) \vee \neg p.$$

Now it is fairly easy to see how to finish the proof: we can switch the $\vee$ statement around using commutativity and simplify using double negation. We can now write down the proof sequence.

Statements	Reasons
1. $p \to q$	given
2. $\neg p \vee q$	implication
3. $q \vee \neg p$	commutativity
4. $\neg(\neg q) \vee \neg p$	double negation
5. $\neg q \to \neg p$	implication

We used the forward–backward approach to move forward from step 1 to step 2, and again to move backward from step 5 to step 4. Then we connected step 2 to step 4 with a simple proof sequence. ◇

You may have noticed that in Section 1.1, we proved the stronger statement

$$p \to q \;\Leftrightarrow\; \neg q \to \neg p$$

using truth tables; the above example proves only the "$\Rightarrow$" direction of this equivalence. To prove the other direction, we need another proof sequence. However, in this case, this other proof sequence is easy to write down, because all of the derivation rules we used were reversible. Implication, commutativity, and double negation are all equivalence rules, so we could write down a new proof sequence with the order of the steps reversed, and we would have a valid proof of the "$\Leftarrow$" direction.

Activity 1.2.2: Proof Sequences

1. Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to.

Statements	Reasons
1. $\neg[(p \wedge \neg q) \to r]$	given
2. $\neg[\neg(p \wedge \neg q) \vee r]$	
3. $\neg[(\neg p \vee \neg\neg q) \vee r]$	
4. $\neg[(\neg p \vee q) \vee r]$	
5. $\neg(\neg p \vee q) \wedge \neg r$	
6. $(\neg\neg p \wedge \neg q) \wedge \neg r$	
7. $(p \wedge \neg q) \wedge \neg r$	
8. $p \wedge (\neg q \wedge \neg r)$	
9. p	

2. Notice that the "given" in step 1 of the above proof sequence is the *negation* of formal logic translation of the pool policy from Activity 1.1.2. In other words, we start by assuming that the pool policy has been violated. Interpret what the conclusion of this sequence tells you in context by completing the sentence: "If you violated the pool policy, then ________."

Exercises 1.2

1. Use truth tables to establish the *modus tollens* tautology:

$$\left.\begin{array}{r}\neg q\\ p \to q\end{array}\right\} \Rightarrow \neg p$$

2. Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to.

Statements	Reasons
1. $q \wedge r$	given
2. $\neg(\neg p \wedge q)$	given
3. $\neg\neg p \vee \neg q$	
4. $p \vee \neg q$	
5. $\neg q \vee p$	
6. $q \to p$	
7. q	
8. p	

3. Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to.

Statements	Reasons
1. $(p \wedge q) \to r$	given
2. $\neg(p \wedge q) \vee r$	
3. $(\neg p \vee \neg q) \vee r$	
4. $\neg p \vee (\neg q \vee r)$	
5. $p \to (\neg q \vee r)$	

4. Is the proof in Exercise 2 reversible? Why or why not?
5. Is the proof in Exercise 3 reversible? Why or why not?
6. Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to.

Statements	Reasons
1. $p \wedge (q \vee r)$	given
2. $\neg(p \wedge q)$	given
3. $\neg p \vee \neg q$	
4. $\neg q \vee \neg p$	
5. $q \to \neg p$	
6. p	
7. $\neg(\neg p)$	
8. $\neg q$	
9. $(q \vee r) \wedge p$	
10. $q \vee r$	
11. $r \vee q$	
12. $\neg(\neg r) \vee q$	
13. $\neg r \to q$	
14. $\neg(\neg r)$	
15. r	
16. $p \wedge r$	

7. Justify each conclusion with a derivation rule.
   (a) If Joe is artistic, he must also be creative. Joe is not creative. Therefore, Joe is not artistic.
   (b) Lingli is both athletic and intelligent. Therefore, Lingli is athletic.
   (c) If Monique is 18 years old, then she may vote. Monique is 18 years old. Therefore, Monique may vote.
   (d) Marianne has never been north of Saskatoon or south of Santo Domingo. In other words, she has never been north of Saskatoon and she has never been south of Santo Domingo.
8. Which derivation rule justifies the following argument?

   If n is a multiple of 4, then n is even. However, n is not even. Therefore, n is not a multiple of 4.

9. Let x and y be integers. Given the statement

   $x > y$ or x is odd.

   what statement follows by the implication rule?
10. Let Q be a quadrilateral. Given the statements

    If Q is a rhombus, then Q is a parallelogram.
    Q is not a parallelogram.

    what statement follows by *modus tollens*?
11. Let x and y be numbers. Simplify the following statement using De Morgan's laws and double negation.

    It is not the case that x is not greater than 3 and y is not found.

12. Write a statement that follows from the statement

    It is sunny and warm today.

    by the simplification rule.
13. Write a statement that follows from the statement

    This soup tastes funny.

    by the addition rule.
14. Recall Exercise 31 of Section 1.1. Suppose that all of the following status reports are correct:
    - Processor B is not working and processor C is working.
    - Processor A is working if and only if processor B is working.
    - At least one of the two processors A and B is not working.

    Let $a =$ "A is working," $b =$ "B is working," and $c =$ "C is working."
    (a) If you haven't already done so, write each status report in terms of a, b, and c, using the symbols of formal logic.
    (b) How would you justify the conclusion that B is not working? (In other words, given the statements in part (a), which derivation rule allows you to conclude $\neg b$?)
    (c) How would you justify the conclusion that C is working?
    (d) Write a proof sequence to conclude that A is not working. (In other words, given the statements in part (a), write a proof sequence to conclude $\neg a$.)
15. Write a proof sequence for the following assertion. Justify each step.

$$\left.\begin{array}{r} p \to \neg q \\ r \to (p \wedge q) \end{array}\right\} \Rightarrow \neg r$$

16. Write a proof sequence for the following assertion. Justify each step.

$$\left.\begin{array}{c} p \\ p \to r \\ q \to \neg r \end{array}\right\} \Rightarrow \neg q$$

17. Write a proof sequence for the following assertion. Justify each step.

$$\left.\begin{array}{c} p \to q \\ p \wedge r \end{array}\right\} \Rightarrow q \wedge r$$

18. Write a proof sequence for the following assertion. Justify one of the steps in your proof using the result of Example 1.8.

$$\left.\begin{array}{c} \neg(a \wedge \neg b) \\ \neg b \end{array}\right\} \Rightarrow \neg a$$

19. Write a proof sequence to establish that $p \Leftrightarrow p \wedge p$ is a tautology.
20. Write a proof sequence to establish that $p \Leftrightarrow p \vee p$ is a tautology. (Hint: Use De Morgan's laws and Exercise 19.)
21. Write a proof sequence for the following assertion. Justify each step.

$$\neg(\neg p \to q) \vee (\neg p \wedge \neg q) \quad \Rightarrow \quad \neg p \wedge \neg q$$

22. Write a proof sequence for the following assertion. Justify each step.

$$(p \vee q) \vee (p \vee r) \quad \Rightarrow \quad \neg r \to (p \vee q)$$

23. Consider the following assertion.

$$\neg(\neg p \vee q) \quad \Rightarrow \quad p \vee q$$

   (a) Find a statement that is one step forward from the given.
   (b) Find a statement that is one step backward from the goal. (Use the addition rule—in reverse—to find a statement from which the goal will follow.)
   (c) Give a proof sequence for the assertion.
   (d) Is your proof reversible? Why or why not?

24. Use a truth table to show that

$$\left.\begin{array}{c} p \to q \\ \neg p \end{array}\right\} \stackrel{?}{\Rightarrow} \neg q$$

is not a tautology. (This example shows that substitution isn't valid for inference rules, in general. Substituting the weaker statement, q, for the stronger statement, p, in the expression "$\neg p$" doesn't work.)

25. (a) Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to.

Statements	Reasons
1. $p \to (q \to r)$	given
2. $\neg p \vee (q \to r)$	
3. $\neg p \vee (\neg q \vee r)$	
4. $(\neg p \vee \neg q) \vee r$	
5. $\neg(p \wedge q) \vee r$	
6. $(p \wedge q) \to r$	

   (b) Explain why the proof in part (a) is reversible.
   (c) The proof in part (a) (along with its reverse) establishes the following tautology:

$$p \to (q \to r) \quad \Leftrightarrow \quad (p \wedge q) \to r$$

Therefore, to prove an assertion of the form $A \Rightarrow B \to C$, it is sufficient to prove

$$\left.\begin{array}{c} A \\ B \end{array}\right\} \Rightarrow C$$

instead. Use this fact to rewrite the tautology

$$p \wedge (q \to r) \;\Rightarrow\; q \to (p \wedge r)$$

as a tautology of the form

$$\left.\begin{array}{c} A \\ B \end{array}\right\} \Rightarrow C,$$

where C does not contain the $\to$ connective. (The process of rewriting a tautology this way is called the *deduction method*.)

(d) Give a proof sequence for the rewritten tautology in part (c).

26. This exercise will lead you through a proof of the *distributive property* of $\wedge$ over $\vee$. We will prove:

$$p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee (p \wedge r).$$

(a) The above assertion is the same as the following:

$$p \wedge (q \vee r) \Rightarrow \neg(p \wedge q) \to (p \wedge r).$$

Why?

(b) Use the deduction method from Exercise 25(c) to rewrite the tautology from part (a).

(c) Prove your rewritten tautology.

27. Use a truth table to show that $(a \to b) \wedge (a \wedge \neg b)$ is a contradiction.

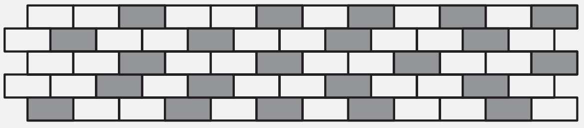
28. Is $a \to \neg a$ a contradiction? Why or why not?

1.3 Predicate Logic

When we defined statements, we said that a sentence of the form "x is even" is not a statement, because its T/F value depends on x. Mathematical writing, however, almost always deals with sentences of this type; we often express mathematical ideas in terms of some unknown variable. This section explains how to extend our formal system of logic to deal with this situation.

Preview 1.3

1. The diagram below shows a standard brick pattern (a "running bond" pattern) composed of two different colors of bricks. The bricklayer had certain rules in mind governing the arrangement of the colors. Devise some possible rules, written as logical statements. Your statements should be as specific as possible but should also hold true for every brick in the pattern.



2. Nikola bets you \$5 that every player on his basketball team will score a point or earn an assist in tonight's game. What must happen for you to win the bet? Express this condition in the simplest, most natural way possible, and explain your reasoning.

3. For each of the following statements, give a list of natural numbers that satisfies the statement. Can you find a single list that satisfies both statements?

   Statement p: There is a number in the list that is greater than every other number in the list.

   Statement q: Every number in the list is less than some other number in the list.

1.3.1 Predicates

Definition 1.3 A *predicate* is a declarative sentence whose T/F value depends on one or more variables. In other words, a predicate is a declarative sentence with variables, and after those variables have been given specific values the sentence becomes a statement.

We use function notation to denote predicates. For example,

$$P(x) = \text{“}x \text{ is even,” and}$$
$$Q(x,y) = \text{“}x \text{ is heavier than } y\text{”}$$

are predicates. The statement $P(8)$ is true, while the statement $Q(\text{feather, brick})$ is false.

Implicit in a predicate is the *domain* (or *universe*) of values that the variable(s) can take. For $P(x)$, the domain could be the integers; for $Q(x,y)$, the domain could be some collection of physical objects. We will usually state the domain along with the predicate, unless it is clear from the context.

Equations are predicates. For example, if $E(x)$ stands for the equation $x^2 - x - 6 = 0$, then $E(3)$ is true and $E(4)$ is false. We regard equations as declarative sentences, where the $=$ sign plays the role of a verb.

1.3.2 Quantifiers

By themselves, predicates aren't statements because they contain free variables. We can make them into statements by plugging in specific values of the domain, but often we would like to describe a range of values for the variables in a predicate. A *quantifier* modifies a predicate by describing whether some or all elements of the domain satisfy the predicate.

We will need only two quantifiers: universal and existential. The *universal quantifier* “for all” is denoted by $\forall$. So the statement

$$(\forall x)P(x)$$

says that $P(x)$ is true *for all* x in the domain. The *existential quantifier* “there exists” is denoted by $\exists$. The statement

$$(\exists x)P(x)$$

says that *there exists* an element x of the domain such that $P(x)$ is true; in other words, $P(x)$ is true *for some* x in the domain.

For example, if $E(x)$ is the real number equation $x^2 - x - 6 = 0$, then the expression $(\exists x)E(x)$ says, “There is some real number x such that $x^2 - x - 6 = 0$,” or more simply, “The equation $x^2 - x - 6 = 0$ has a solution.” The variable x is no longer a free variable, since the $\exists$ quantifier changes the role it plays in the sentence.

If $Z(x)$ represents the real number equation $x \cdot 0 = 0$, the expression $(\forall x)Z(x)$ means “For all real numbers x, $x \cdot 0 = 0$.” Again, this is a sentence without free variables, since the range of possible values for x is clearly specified.

When we put a quantifier in front of a predicate, we form a *quantified statement*. Since the quantifier restricts the range of values for the variables in the predicate, the quantified statement is either true or false (but not both). In the above examples, $(\exists x)E(x)$ and $(\forall x)Z(x)$ are both true, while the statement $(\forall x)E(x)$ is false, since there are some real numbers that do not satisfy the equation $x^2 - x - 6 = 0$.

The real power of predicate logic comes from combining quantifiers, predicates, and the symbols of propositional logic. For example, if we would like to claim that there is a negative number that satisfies the equation $x^2 - x - 6 = 0$, we could define a new predicate

$$N(x) = \text{“}x \text{ is negative.”}$$

Then the statement

$$(\exists x)(N(x) \wedge E(x))$$

translates as "There exists some real number x such that x is negative and $x^2 - x - 6 = 0$."

The *scope* of a quantifier is the part of the formula to which the quantifier refers. In a complicated formula in predicate logic, it is important to use parentheses to indicate the scope of each quantifier. In general, the scope is what lies inside the set of parentheses right after the quantifier:

$$(\forall x)(\ldots \text{scope of } \forall \ldots), \qquad (\exists x)(\ldots \text{scope of } \exists \ldots).$$

In the statement $(\exists x)(N(x) \wedge E(x))$, the scope of the $\exists$ quantifier is the expression $N(x) \wedge E(x)$.

1.3.3 Translation

There are lots of different ways to write quantified statements in English. Translating back and forth between English statements and predicate logic is a skill that takes practice.

Example 1.10 Using all cars as a domain, if

$$P(x) = \text{“}x \text{ gets good mileage.”}$$
$$Q(x) = \text{“}x \text{ is large.”}$$

then the statement $(\forall x)(Q(x) \to \neg P(x))$ could be translated very literally as

> "For all cars x, if x is large, then x does not get good mileage."

However, a more natural translation of the same statement is

> "All large cars get bad mileage."

or

> "There aren't any large cars that get good mileage."

If we wanted to say the opposite—that is, that there are some large cars that get good mileage—we could write the following.

$$(\exists x)(P(x) \wedge Q(x))$$

We'll give a formal proof that this negation is correct in Example 1.13.

The next example shows how a seemingly simple mathematical statement yields a rather complicated formula in predicate logic. The careful use of predicates can help reveal the logical structure of a mathematical claim.

Example 1.11 In the domain of all integers, let $P(x) =$ "x is even." We can express the fact that the sum of an even number with an odd number is odd as follows.

$$(\forall x)(\forall y)[(P(x) \wedge \neg P(y)) \to (\neg P(x+y))]$$

Of course, the literal translation of this quantified statement is "For all integers x and for all integers y, if x is even and y is not even, then $x+y$ is not even," but we normally say something informal like "An even plus an odd is odd."

This last example used two universal quantifiers to express a fact about an arbitrary pair x, y of integers. The next example shows what can happen when you combine universal and existential quantifiers in the same statement.

Example 1.12 In the domain of all real numbers, let $G(x, y)$ be the predicate "$x > y$." The statement

$$(\forall y)(\exists x)G(x, y)$$

says literally that "For all numbers y, there exists some number x such that $x > y$," or more simply, "Given any number y, there is some number that is greater than y." This statement is clearly true: the number $y + 1$ is always greater than y, for example. However, the statement

$$(\exists x)(\forall y)G(x,y)$$

translates literally as "There exists a number x such that, for all numbers y, $x > y$." In simpler language, this statement says, "There is some number that is greater than any other number." This statement is clearly false, because there is no largest number.

The order of the quantifiers matters. In both of these statements, a claim is made that x is greater than y. In the first statement, you are first given an arbitrary number y, then the claim is that it is possible to find some x that is greater than it. However, the second statement claims there is some number x, such that, given any other y, x will be the greater number. In the second statement, you must decide on what x is before you pick y. In the first statement, you pick y first, then you can decide on x.

Activity 1.3.1: Translating Quantified Statements

1. In the domain of natural numbers, let $P(x)$ be the statement "x is even" and let $Q(x)$ be the statement "x is a multiple of 10." Translate the following statements into natural English statements, and decide which of them are true.
   (a) $(\forall x)(P(x) \to Q(x))$
   (b) $(\forall x)(Q(x) \to P(x))$
   (c) $(\exists x)(Q(x) \wedge P(x))$
2. Let $P(x,y)$ be the statement "$x^2 + y = 7$" in the domain of all real numbers.
   (a) When using more than one quantifier in a formula, we read them from left to right, and we don't always write all the parentheses. For example, the expression $(\forall y)(\exists x)P(x,y)$ means $(\forall y)[(\exists x)P(x,y)]$, translated as, "For all y, there is some x such that $x^2 + y = 7$." Decide whether each of the following expressions is true or false. Give a brief justification for each answer.
      i $(\forall y)(\exists x)P(x,y)$
      ii $(\forall x)(\exists y)P(x,y)$
      iii $(\exists y)(\forall x)P(x,y)$
      iv $(\exists x)(\exists y)P(x,y)$
      v $(\exists y)(\exists x)P(x,y)$
      vi $(\forall x)(\forall y)P(x,y)$
      vii $(\forall y)(\forall x)P(x,y)$
   (b) Based on these examples, when do you think it is permissible to switch the order of quantifiers?

1.3.4 Negation

In mathematical arguments, it is often important to be able to accurately express the logical opposite, or *negation* of a quantified statement. The next activity encourages you to discover some formal rules for doing so.

Activity 1.3.2: Negating Quantified Statements

1. Let $H(x)$ be the statement "x is happy," where the domain is the set of dogs.
   (a) How would you write the statement "all dogs are happy" in the symbols of predicate logic?
   (b) The **negation** of a statement is its logical opposite; it is true when the statement is false, and false when the statement is true. For each of the following statements, decide whether the statement correctly expresses the negation of the statement "all dogs are happy."

i All dogs are not happy.
ii Not all dogs are happy.
iii Some dogs are not happy.

(c) For each statement in (b) that correctly expresses the negation of the statement "all dogs are happy," write the statement in the symbols of predicate logic.

2. Let X be the set of real numbers in the interval $0 < x \leq 1$. Consider the following statement.

For every number x in X, there is a number y in X such that $y < x$.

(a) Decide whether this statement is true or false.
(b) Write the negation of this statement in English.
(c) Write the above statement, and its negation, in the symbols of predicate logic.

3. Based on your work above, state some symbolic rules you can use to negate a quantified statement.

The preceding activity suggests the following *negation rules* for predicate logic.

Equivalence	Name
$\neg[(\forall x)P(x)] \Leftrightarrow (\exists x)(\neg P(x))$	universal negation
$\neg[(\exists x)P(x)] \Leftrightarrow (\forall x)(\neg P(x))$	existential negation

It is easy to see the pattern of these two rules: to negate a quantified statement, bring the negation inside the quantifier, and switch the quantifier.

Let's interpret the negation rules in the context of an example. In the domain of all people, let $L(x)$ stand for "x is a liar." The universal negation rule says that the negation of "All people are liars" is "There exists a person who is not a liar." In symbols,

$$\neg[(\forall x)L(x)] \Leftrightarrow (\exists x)(\neg L(x)).$$

Similarly, the existential negation rule says that the negation of "There exists a liar" is "There are no liars."

Example 1.13 In Example 1.10, we discussed what the negation of the statement

"All large cars get bad mileage."

should be. We can answer this question by negating the formal statement

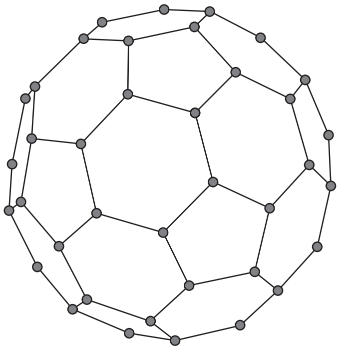
$$(\forall x)(Q(x) \to \neg P(x)) \tag{1.3.1}$$

using a proof sequence. We'll suppose as given the negation of statement 1.3.1, and deduce an equivalent statement.

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \to \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \to \neg P(x))$	universal negation
3. $(\exists x)\neg(\neg Q(x) \vee \neg P(x))$	implication
4. $(\exists x)(\neg(\neg Q(x)) \wedge \neg(\neg P(x)))$	De Morgan's law
5. $(\exists x)(Q(x) \wedge P(x))$	double negation
6. $(\exists x)(P(x) \wedge Q(x))$	commutativity

Notice that the result of our formal argument agrees with the intuitive negation we did in Example 1.10: There exists some car that is both large and gets good mileage.

Example 1.14 Let the domain be all faces of the following truncated icosahedron (also known as a soccer ball).



Consider the following predicates:

$$P(x) = \text{“}x \text{ is a pentagon.”}$$
$$H(x) = \text{“}x \text{ is a hexagon.”}$$
$$B(x,y) = \text{“}x \text{ borders } y\text{.”}$$

Here we say that two polygons border each other if they share an edge. We also stipulate that a polygon cannot border itself. Confirm that the following observations are true for any truncated icosahedron.

1. No two pentagons border each other.
2. Every pentagon borders some hexagon.
3. Every hexagon borders another hexagon.

Write these statements in predicate logic and negate them. Simplify the negated statements so that no quantifier or connective lies within the scope of a negation. Translate your negated statement back into English.

Solution: The formalizations of these statements are as follows.

1. $(\forall x)(\forall y)((P(x) \wedge P(y)) \to \neg B(x,y))$
2. $(\forall x)(P(x) \to (\exists y)(H(y) \wedge B(x,y)))$
3. $(\forall x)(H(x) \to (\exists y)(H(y) \wedge B(x,y)))$

We'll negate (2), and leave the others as exercises. See if you can figure out the reasons for each equivalence.

$$\begin{aligned}
\neg[(\forall x)(P(x) \to (\exists y)(H(y) \wedge B(x,y)))] &\Leftrightarrow (\exists x)[\neg(P(x) \to (\exists y)(H(y) \wedge B(x,y)))] \\
&\Leftrightarrow (\exists x)[\neg(\neg P(x) \vee (\exists y)(H(y) \wedge B(x,y)))] \\
&\Leftrightarrow (\exists x)[\neg\neg P(x) \wedge \neg(\exists y)(H(y) \wedge B(x,y))] \\
&\Leftrightarrow (\exists x)[\neg\neg P(x) \wedge (\forall y)\neg(H(y) \wedge B(x,y))] \\
&\Leftrightarrow (\exists x)(P(x) \wedge (\forall y)\neg(H(y) \wedge B(x,y))) \\
&\Leftrightarrow (\exists x)(P(x) \wedge (\forall y)(\neg H(y) \vee \neg B(x,y))) \\
&\Leftrightarrow (\exists x)(P(x) \wedge (\forall y)(H(y) \to \neg B(x,y)))
\end{aligned}$$

This last statement says that there exists an x such that x is a pentagon and, for any y, if y is a hexagon, then x does not border y. In other words, there is some pentagon that borders no hexagon. If you found a solid with this property, it couldn't be a truncated icosahedron. ◇

1.3.5 Two Common Constructions

There are two expressions that come up often, and knowing the predicate logic for these expressions makes translation much easier. The first is the statement

All ⟨blanks⟩ are ⟨something⟩.

For example, “All baseball players are rich,” or “All oysters taste funny.” In general, if $P(x)$ and $Q(x)$ are the predicates “x is ⟨blank⟩” and “x is ⟨something⟩,” respectively, then the predicate logic expression

$$(\forall x)(P(x) \to Q(x))$$

translates as “For all x, if x is ⟨blank⟩, then x is ⟨something⟩.” Put more simply, “All x’s with property ⟨blank⟩ must have property ⟨something⟩,” or even simpler, “All ⟨blanks⟩ are ⟨something⟩.” In the domain of all people, if $R(x)$ stands for “x is rich” and $B(x)$ stands for “x is a baseball player,” then $(\forall x)(B(x) \to R(x))$ is the statement “All baseball players are rich.”

The second construction is of the form

There is a ⟨blank⟩ that is ⟨something⟩.

For example, “There is a rich baseball player,” or “There is a funny-tasting oyster.” This expression has the following form in predicate logic.

$$(\exists x)(P(x) \wedge Q(x))$$

Note that this translates literally as “There is some x such that x is ⟨blank⟩ and x is ⟨something⟩,” which is what we want. In the domain of shellfish, if $O(x)$ is the predicate “x is an oyster” and $F(x)$ is the predicate “x tastes funny,” then $(\exists x)(F(x) \wedge O(x))$ would translate as “There is a funny-tasting oyster.” Note that you could also say “There is an oyster that tastes funny,” “Some oysters taste funny,” or, more awkwardly, “There is a funny-tasting shellfish that is an oyster.” These statements all mean the same thing.

Exercises 1.3

1. In the domain of integers, let $P(x, y)$ be the predicate “$x \cdot y = 12$.” Tell whether each of the following statements is true or false.
   (a) $P(3, 4)$
   (b) $P(3, 5)$
   (c) $P(2, 6) \vee P(3, 7)$
   (d) $(\forall x)(\forall y)(P(x, y) \to P(y, x))$
   (e) $(\forall x)(\exists y)P(x, y)$
2. In the domain of all penguins, let $D(x)$ be the predicate “x is dangerous.” Translate the following quantified statements into simple, everyday English.
   (a) $(\forall x)D(x)$
   (b) $(\exists x)D(x)$
   (c) $\neg(\exists x)D(x)$
   (d) $(\exists x)\neg D(x)$
3. In the domain of all movies, let $V(x)$ be the predicate “x is violent.” Write the following statements in the symbols of predicate logic.
   (a) Some movies are violent.
   (b) Some movies are not violent.
   (c) No movies are violent.
   (d) All movies are violent.
4. Let the following predicates be given. The domain is all mammals.

$$L(x) = \text{“}x \text{ is a lion.”}$$
$$F(x) = \text{“}x \text{ is fuzzy.”}$$

Translate the following statements into predicate logic.

(a) All lions are fuzzy.

(b) Some lions are fuzzy.

5. In the domain of all books, consider the following predicates.

$$H(x) = \text{“}x \text{ is heavy.”}$$
$$C(x) = \text{“}x \text{ is confusing.”}$$

Translate the following statements in predicate logic into ordinary English.

(a) $(\forall x)(H(x) \to C(x))$

(b) $(\exists x)(C(x) \wedge H(x))$

(c) $(\forall x)(C(x) \vee H(x))$

(d) $(\exists x)(H(x) \wedge \neg C(x))$

6. The domain of the following predicates is the set of all plants.

$$P(x) = \text{“}x \text{ is poisonous.”}$$
$$Q(x) = \text{“Jeff has eaten } x\text{.”}$$

Translate the following statements into predicate logic.

(a) Some plants are poisonous.

(b) Jeff has never eaten a poisonous plant.

(c) There are some nonpoisonous plants that Jeff has never eaten.

7. In the domain of nonzero integers, let $I(x, y)$ be the predicate “x/y is an integer.” Determine whether the following statements are true or false, and explain why.

(a) $(\forall y)(\exists x)I(x, y)$

(b) $(\exists x)(\forall y)I(x, y)$

8. In the domain of integers, consider the following predicates: Let $N(x)$ be the statement “$x \neq 0$.” Let $P(x, y)$ be the statement “$xy = 1$.”

(a) Translate the following statement into the symbols of predicate logic.

For all integers x, there is some integer y such that if $x \neq 0$, then $xy = 1$.

(b) Write the negation of your answer to part (a) in the symbols of predicate logic. Simplify your answer so that it uses the $\wedge$ connective.

(c) Translate your answer from part (b) into an English sentence.

(d) Which statement, (a) or (b), is true in the domain of integers? Explain.

9. Let $P(x, y, z)$ be the predicate “$x + y = z$.”

(a) Simplify the statement $\neg(\forall x)(\forall y)(\exists z)P(x, y, z)$ so that no quantifier lies within the scope of a negation.

(b) Is the statement $(\forall x)(\forall y)(\exists z)P(x, y, z)$ true in the domain of all integers? Explain why or why not.

(c) Is the statement $(\forall x)(\forall y)(\exists z)P(x, y, z)$ true in the domain of all integers between 1 and 100? Explain why or why not.

10. The domain of the following predicates is the set of all traders who work at the Tokyo Stock Exchange.

$$P(x, y) = \text{“}x \text{ makes more money than } y\text{.”}$$
$$Q(x, y) = \text{“}x \neq y\text{”}$$

Translate the following predicate logic statements into *ordinary, everyday* English. (Don't simply give a word-for-word translation; try to write sentences that make sense.)

(a) $(\forall x)(\exists y)P(x,y)$

(b) $(\exists y)(\forall x)(Q(x,y) \to P(x,y))$

(c) Which statement is impossible in this context? Why?

11. Translate the following statements into predicate logic using the two common constructions in Section 1.3.5. State what your predicates are, along with the domain of each.

(a) All natural numbers are integers.

(b) Some integers are natural numbers.

(c) All the streets in Cozumel, Mexico, are one-way.

(d) Some streets in London don't have modern curb cuts.

12. Write the following statements in predicate logic. Define your predicates. Use the domain of all quadrilaterals.

(a) All rhombuses are parallelograms.

(b) Some parallelograms are not rhombuses.

13. Let the following predicates be given. The domain is all people.

$$R(x) = \text{“}x \text{ is rude.”}$$
$$\neg R(x) = \text{“}x \text{ is pleasant.”}$$
$$C(x) = \text{“}x \text{ is a child.”}$$

(a) Write the following statement in predicate logic.

There is at least one rude child.

(b) Formally negate your statement from part (a).

(c) Write the English translation of your negated statement.

14. In the domain of all people, consider the following predicate.

$$P(x,y) = \text{“}x \text{ needs to love } y\text{.”}$$

(a) Write the statement "Everybody needs somebody to love" in predicate logic.

(b) Formally negate your statement from part (a).

(c) Write the English translation of your negated statement.

15. The domain for this problem is some unspecified collection of numbers. Consider the predicate

$$P(x,y) = \text{“}x \text{ is greater than } y\text{.”}$$

(a) Translate the following statement into predicate logic.

Every number has a number that is greater than it.

(b) Negate your expression from part (a), and simplify it so that no quantifier or connective lies within the scope of a negation.

(c) Translate your expression from part (b) into understandable English. Don't use variables in your English translation.

16. Any equation or inequality with variables in it is a predicate in the domain of real numbers. For each of the following statements, tell whether the statement is true or false.

(a) $(\forall x)(x^2 > x)$

(b) $(\exists x)(x^2 - 2 = 1)$

(c) $(\exists x)(x^2 + 2 = 1)$
(d) $(\forall x)(\exists y)(x^2 + y = 4)$
(e) $(\exists y)(\forall x)(x^2 + y = 4)$

17. The domain of the following predicates is all integers greater than 1.

$$P(x) = \text{“}x \text{ is prime.”}$$
$$Q(x,y) = \text{“}x \text{ divides } y\text{.”}$$

Consider the following statement.

For every x that is not prime, there is some prime y that divides it.

(a) Write the statement in predicate logic.
(b) Formally negate the statement.
(c) Write the English translation of your negated statement.

18. Write the following statement in predicate logic, and negate it. Say what your predicates are, along with the domains.

Let x and y be real numbers. If x is rational and y is irrational, then $x + y$ is irrational.

19. Refer to Example 1.14.
(a) Give the reasons for each $\Leftrightarrow$ step in the simplification of the formal negation of statement (2).
(b) Give the formal negation of statement (1). Simplify your answer so that no quantifier or connective lies within the scope of a negation. Translate your negated statement back into English.
(c) Give the formal negation of statement (3). Simplify your answer. Translate your negated statement back into English.

20. Let the following predicates be given in the domain of all triangles.

$$R(x) = \text{“}x \text{ is a right triangle.”}$$
$$B(x) = \text{“}x \text{ has an obtuse angle.”}$$

Consider the following statements.

$$S_1 = \neg(\exists x)(R(x) \wedge B(x))$$
$$S_2 = (\forall x)(R(x) \rightarrow \neg B(x))$$

(a) Write a proof sequence to show that $S_1 \Leftrightarrow S_2$.
(b) Write S_1 in ordinary English.
(c) Write S_2 in ordinary English.

21. Let the following predicates be given. The domain is all computer science classes.

$$I(x) = \text{“}x \text{ is interesting.”}$$
$$U(x) = \text{“}x \text{ is useful.”}$$
$$H(x,y) = \text{“}x \text{ is harder than } y\text{.”}$$
$$M(x,y) = \text{“}x \text{ has more students than } y\text{.”}$$

(a) Write the following statements in predicate logic.
i All interesting CS classes are useful.

ii There are some useful CS classes that are not interesting.

iii Every interesting CS class has more students than any non-interesting CS class.

(b) Write the following predicate logic statement in everyday English. Don't just give a word-for-word translation; your sentence should make sense.

$$(\exists x)[I(x) \wedge (\forall y)(H(x,y) \rightarrow M(y,x))]$$

(c) Formally negate the statement from part (b). Simplify your negation so that no quantifier lies within the scope of a negation. State which derivation rules you are using.

(d) Give a translation of your negated statement in everyday English.

22. Let the following predicates be given. The domain is all cars.

$$F(x) = \text{"}x \text{ is fast."}$$
$$S(x) = \text{"}x \text{ is a sports car."}$$
$$E(x) = \text{"}x \text{ is expensive."}$$
$$A(x,y) = \text{"}x \text{ is safer than } y.\text{"}$$

(a) Write the following statements in predicate logic.

i All sports cars are fast.

ii There are fast cars that aren't sports cars.

iii Every fast sports car is expensive.

(b) Write the following predicate logic statement in everyday English. Don't just give a word-for-word translation; your sentence should make sense.

$$(\forall x)[S(x) \rightarrow (\exists y)(E(y) \wedge A(y,x))]$$

(c) Formally negate the statement from part (b). Simplify your negation so that no quantifier or connective lies within the scope of a negation. State which derivation rules you are using.

(d) Give a translation of your negated statement in everyday English.

23. Let $P(x)$ be a predicate in the domain consisting of just the numbers 0 and 1. Let p be the statement $P(0)$ and let q be the statement $P(1)$.

(a) Write $(\forall x)P(x)$ as a propositional logic formula using p and q.

(b) Write $(\exists x)P(x)$ as a propositional logic formula using p and q.

(c) In this situation, which derivation rule from propositional logic corresponds to the universal and existential negation rules of predicate logic?

24. (a) Give an example of a pair of predicates $P(x)$ and $Q(x)$ in some domain to show that the $\exists$ quantifier does not distribute over the $\wedge$ connective. That is, give an example to show that the statements

$$(\exists x)(P(x) \wedge Q(x)) \quad \text{and} \quad (\exists x)P(x) \wedge (\exists x)Q(x)$$

are not logically equivalent.

(b) It is true, however, that $\exists$ distributes over $\vee$. That is,

$$(\exists x)(P(x) \vee Q(x)) \Leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$$

is an equivalence rule for predicate logic. Verify that your example from part (a) satisfies this equivalence.

25. (a) Give an example to show that $\forall$ does not distribute over $\vee$.
    (b) It is a fact that $\forall$ distributes over $\wedge$. Check that your example from part (a) satisfies this equivalence rule.

1.4 Logic in Mathematics

There is much more that we could say about symbolic logic; we have only scratched the surface. But we have developed enough tools to help us think carefully about the types of language mathematicians use. This section provides an overview of the basic mathematical "parts of speech."

Most mathematics textbooks (including this one) label important statements with a heading, such as "Theorem," "Definition," or "Proof." The name of each statement describes the role it plays in the logical development of the subject. Therefore, it is important to understand the meanings of these different statement labels.

Preview 1.4

1. Explain why an integer cannot be both even and odd.
2. Draw a diagram consisting of straight line segments in which every line segment intersects exactly four other line segments.
3. Recall that a *prime number* is a natural number n such that $n > 1$ and n has no divisors other than n and 1. Prove or disprove the following: Every prime number greater than 3 is the sum of two prime numbers.

1.4.1 The Role of Definitions in Mathematics

When we call a statement a "definition" in mathematics, we mean something different from the usual everyday notion. Everyday definitions are *descriptive*. The thing being defined already exists, and the purpose of the definition is to describe the thing. When a dictionary defines some term, it is characterizing the way the term is commonly used. For example, if we looked up the definition of "mortadella" in the *Oxford English Dictionary* (OED), we would read the following.

> Any of several types of Italian (esp. Bolognese) sausage; (now) spec. a thick smooth-textured pork sausage containing pieces of fat and typically served in slices.

The authors of the OED have done their best to describe what is meant by the term "mortadella." A good dictionary definition is one that does a good job describing something.

In mathematics, by contrast, a *definition* is a statement that stipulates the meaning of a new term, symbol, or object. For example, a plane geometry textbook may define parallel lines as follows.

Definition 1.4 Two lines are *parallel* if they have no points in common.

The job of this definition is not to describe parallel lines, but rather to specify exactly what we mean when we use the word "parallel." Once parallel lines have been defined in this way, the statement "l and m are parallel" means "l and m have no points in common." We may have some intuitive idea of what l and m might look like (e.g., they must run in the same direction), but for the purposes of any future arguments, the only thing we really *know* about l and m is that they don't intersect each other.

The meaning of a mathematical statement depends on the definitions of the terms involved. If you don't understand a mathematical statement, start looking at the definitions of all the terms. These definitions stipulate the meanings of the terms. The statement won't make sense without them.

For example, consider preview question 1.4.1 at the beginning of this section. We already know what even and odd numbers are; we all come to this problem with a previously learned *concept image* of "even" and "odd." Our concept image is what we think of when we hear the term: an even number ends in an even digit, an odd number can't be divided in half evenly, "2, 4, 6, 8; who do we appreciate," etc. When writing mathematically, however, it is important not to rely too heavily on these concept images. Any mathematical statement about even and odd numbers derives its meaning from definitions. We choose to specify these as follows.

Definition 1.5 An integer n is *even* if $n = 2k$ for some integer k.

Definition 1.6 An integer n is *odd* if $n = 2k + 1$ for some integer k.

Given these definitions, we can justify the statement "17 is odd" by noting that $17 = 2 \cdot 8 + 1$. In fact, this equation is precisely the meaning of the statement that "17 is odd"; there is some integer k (in this case, $k = 8$) such that $17 = 2k + 1$. You already "knew" that 17 is odd, but in order to mathematically *prove* that 17 is odd, you need to use the definition.

Mathematical definitions must be extremely precise, and this can make them somewhat limited. Often, our concept image contains much more information than the definition supplies. For example, we probably all agree that it is impossible for a number to be both even and odd, but this fact doesn't follow immediately from Definitions 1.5 and 1.6. To say that some given number n is even means that $n = 2k_1$ for some integer k_1, and to say that it is odd is to say that $n = 2k_2 + 1$ for some integer k_2. (Note that k_1 and k_2 may be different.) Now, is this possible? It would imply that $2k_1 = 2k_2 + 1$, which says that $1 = 2(k_1 - k_2)$, showing that 1 is even, by Definition 1.5. At this point we might object that 1 is odd, so it can't be even, but this reasoning is circular: we were trying to show that a number cannot be both even and odd. We haven't yet shown this fact, so we can't use this fact in our argument. It turns out that Definitions 1.5 and 1.6 alone are not enough to show that a number can't be both even and odd; to do so requires more facts about integers, as we will see in Section 1.5.

One reasonable objection to the above discussion is that our definition of odd integers was too limiting; why not define an odd integer to be an integer that isn't even? This is certainly permissible, but then it would be hard[2] to show that an odd integer n can be written as $2k + 1$ for some integer k. And we can't have two definitions for the same term. Stipulating a definition usually involves a choice on the part of the author, but once this choice is made, we are stuck with it. We have chosen to define odd integers as in Definition 1.6, so this is what we mean when we say "odd."

Since definitions are stipulative, they are logically "if and only if" statements. However, it is common to write definitions in the form

[Object] x is [defined term] if [defining property about x].

The foregoing examples all take this form. In predicate logic, if

$$D(x) = x \text{ is [defined term]}$$
$$P(x) = \text{[defining property about } x\text{]}$$

then the above definition really means $(\forall x)(P(x) \leftrightarrow D(x))$. However, this is not what the definition says at face value. Definitions look like "if ... then" statements, but we interpret them as "if and only if" statements because they are definitions. For example, Definition 1.4 is stipulating the property that defines all parallel lines, not just a property some parallel lines might have. Strictly speaking, we really should use "if and only if" instead of "if" in our definitions. But the use of "if" is so widespread that most mathematicians would find a definition like

Two lines are *parallel* if and only if they have no points in common.

awkward to read. Since this statement is a definition, it is redundant to say "if and only if."

1.4.2 Other Types of Mathematical Statements

Definitions are a crucial part of mathematics, but there are other kinds of statements that occur frequently in mathematical writing. Any mathematical system needs to start with some assumptions. Without any statements to build on, we would never be able to prove any new statements. Statements that are assumed without proof are called *postulates* or *axioms*. For example, the following is a standard axiom about the natural numbers.

If n is a natural number, so is $n + 1$.

Axioms are typically very basic, fundamental statements about the objects they describe. Any theorem in mathematics is based on the assumption of some set of underlying axioms. So to say theorems are "true" is not to say they are true in any absolute sense, only that they are true, given that some specified set of axioms is true.

2. Actually, it would be impossible, without further information.

A *theorem* is a statement that follows logically from statements we have already established or taken as given. Before a statement can be called a theorem, we must be able to prove it. A *proof* is a valid argument, based on axioms, definitions, and proven theorems, that demonstrates the truth of a statement. The derivation sequences that we did in Section 1.2 were very basic mathematical proofs. We will see more interesting examples of proofs in the next section.

We also use the terms *lemma*, *proposition*, and *corollary* to refer to specific kinds of theorems. Usually authors will label a result a lemma if they are using it to prove another result. Some authors make no distinction between a theorem and a proposition, but the latter often refers to a result that is perhaps not as significant as a full-fledged theorem. A corollary is a theorem that follows immediately from another result via a short argument.

One last word on terminology: A statement that we intend to prove is called a *claim*. A statement that we can't yet prove but that we suspect is true is called a *conjecture*.

1.4.3 Counterexamples

Often, mathematical statements are of the form

$$(\forall x)P(x). \tag{1.4.1}$$

We saw in the previous section that the negation of statement 1.4.1 is

$$(\exists x)\neg P(x). \tag{1.4.2}$$

So either statement 1.4.1 is true or statement 1.4.2 is true, but not both. If we can find a single value for x that makes $\neg P(x)$ true, then we know that statement 1.4.2 is true, and therefore we also know that statement 1.4.1 is false.

For example, we might be tempted to make the following statement.

$$\text{Every prime number is odd.} \tag{1.4.3}$$

But 2 is an example of a prime number that is not odd, so statement 1.4.3 is false. A particular value that shows a statement to be false is called a *counterexample* to the statement.

Another common logical form in mathematics is the universal if–then statement, $(\forall x)(P(x) \to Q(x))$. To find a counterexample to a statement of this form, we need to find some x that satisfies its negation, $(\exists x)\neg(P(x) \to Q(x))$. This last statement is equivalent (using implication and De Morgan's law) to $(\exists x)(P(x) \wedge \neg Q(x))$. So a counterexample is something that satisfies P and violates Q.

Example 1.15 Find a counterexample to the following statement.

For all sequences of numbers $a_1, a_2, a_3, \ldots$, if $a_1 < a_2 < a_3 < \cdots$, then some a_i must be positive.

Solution: By the above discussion, we need an example of a sequence that satisfies the "if" part of the statement and violates the "then" part. In other words, we need to find an increasing sequence that is always negative. Something with a horizontal asymptote will work: $a_n = -1/n$ is one example. Note that $-1 < -1/2 < -1/3 < \cdots$, but all the terms are less than zero. ◇

1.4.4 Axiomatic Systems

Activity 1.4.1: Undefined Terms and Axioms

An axiomatic system is a collection of words, called *undefined terms*, and rules, called *axioms*. The properties of the undefined terms are stipulated by the axioms. Here's an example.

Undefined terms: container, object, contain

Axioms:

1. Every container contains exactly two objects.
2. Every pair of objects is contained in exactly one container.
3. There are exactly four objects.

1. Draw a picture of objects and containers such that all three axioms are satisfied. You will have to decide what an object and a container look like, and what it looks like for a container to contain an object.
2. How many containers did you have in your picture in part (1)? Could you have predicted this from the axioms?
3. Write the negation of Axiom 1 in English in the form "Some container ______________."
4. Draw a picture of objects and containers such that Axioms 2 and 3 are satisfied, but Axiom 1 is not.
5. If you only have to satisfy Axioms 2 and 3, is there any limit on the number of containers you could have?

In rigorous, modern treatments of mathematics, any system (e.g., plane geometry, the real numbers) must be clearly and unambiguously defined from the start. The definitions should leave nothing to intuition; they mean what they say and nothing more. It is important to be clear about the assumptions, or axioms, for the system. Every theorem in the system must be proved with a valid argument, using only the definitions, axioms, and previously proved theorems of the system.

This sounds good, but it is actually impossible. It is impossible because we can't define everything; before we write the first definition we have to have some words in our vocabulary. These starting words are called *undefined terms*. An undefined term has no meaning—it is an abstraction. Its meaning comes from the role it plays in the axioms of the system. A collection of undefined terms and axioms is called an *axiomatic system*.

Axiomatic systems for familiar mathematics such as plane geometry and the real number system are actually quite complicated and beyond the scope of an introductory course. Here we will look at some very simple axiomatic systems to get a feel for how they work. This will also give us some experience with logic in mathematics.

The first example defines a "finite geometry," that is, a system for geometry with a finite number of points. Although this system speaks of "points" and "lines," these terms don't mean the same thing they meant in high school geometry. In fact, these terms don't mean anything at all, to begin with at least. The only thing we know about points and lines is that they satisfy the given axioms.

Example 1.16 Axiomatic system for a four-point geometry.

Undefined terms: point, line, is on

Axioms:

1. For every pair of distinct points x and y, there is a unique line l such that x is on l and y is on l.
2. Given a line l and a point x that is not on l, there is a unique line m such that x is on m and no point on l is also on m.
3. There are exactly four points.
4. It is impossible for three points to be on the same line.

Notice that these axioms use terms from logic in addition to the undefined terms. We are also using numbers ("four" and "three"), even though we haven't defined an axiomatic system for the natural numbers. In this case, our use of numbers is more a convenient shorthand than anything; we aren't relying on any properties of the natural numbers such as addition, ordering, divisibility, etc.

It is common to use an existing system to define a new axiomatic system. For example, some modern treatments of plane geometry use axioms that rely on the real number system. The axioms in Example 1.16 use constructions from predicate logic. In any event, these prerequisite systems can also be defined axiomatically, so systems that use them are still fundamentally axiomatic.

Definitions can help make an axiomatic system more user-friendly. In the four-point geometry of Example 1.16, we could make the following definitions. In these (and other) definitions, the word being defined is in *italics*.

Definition 1.7 A line l *passes through* a point x if x is on l.

Definition 1.7 gives us a convenient alternative to using the undefined term "is on." For example, in the first axiom, it is a bit awkward to say "x is on l and y is on l," but Definition 1.7 allows us to rephrase this as "l passes through x and y." The definition doesn't add any new features to the system; it just helps us describe things more easily. This is basically what any definition in mathematics does. The following definition is a slight restatement of Definition 1.4, modified to fit the terminology of this system.

Definition 1.8 Two lines, l and m, are *parallel* if there is no point x, such that x is on l and x is on m.

Now we could rephrase the second axiom of Example 1.16 as follows.

2. Given a line l and a point x that is not on l, there is a unique line m passing through x such that m is parallel to l.

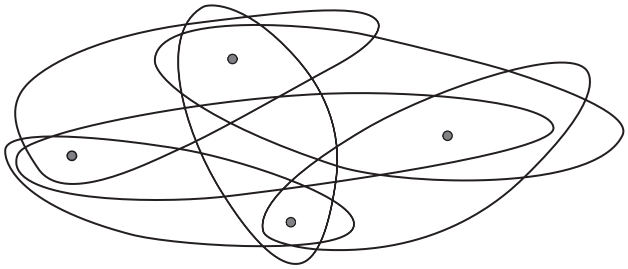
A simple theorem and proof would appear as follows.

Theorem 1.1 *In the axiomatic system of Example 1.16, there are at least two distinct lines.*

Proof By Axiom 3, there are distinct points x, y, and z. By Axiom 1, there is a line l_1 through x and y, and a line l_2 through y and z. By Axiom 4, x, y, and z are not on the same line, so l_1 and l_2 must be distinct lines. □

A *model* of an axiomatic system is an interpretation in some context in which all the undefined terms have meanings and all the axioms hold. Models are important because they show that it is possible for all the axioms to be true, at least in some context. And any theorem that follows from the axioms must also be true for any valid model.

Let's make a model for the system in Example 1.16. Let a "point" be a dot, and let a "line" be a simple closed loop. A point "is on" a line if the dot is inside the loop:



It is easy to check that all the axioms hold, though this model doesn't really match our concept image of points and lines in ordinary geometry. We may think we know what points and lines should look like, but mathematically speaking we only know whatever we can prove about them using the axioms. (In the exercises you will construct a more intuitive model for this system.)

The mathematician David Hilbert (1862–1943) was largely responsible for developing the modern approach to axiomatics. Hilbert, reflecting on the abstract nature of axiomatic systems, remarked, "Instead of points, lines, and planes, one must be able to say at all times tables, chairs, and beer mugs" [24]. If we used a word processor to replace every occurrence of "point" with "table" and every occurrence of "line" with "chair" in the above axioms, definitions, theorem, and proof, the theorem would still hold, and the proof would still be valid.

The following activity will give you some practice thinking abstractly about axiomatic systems. The choice of the words used for undefined terms emphasizes that these terms, by themselves, carry no meaning.

Activity 1.4.2: Models for Axiomatic Systems

Consider the zork-gork-snork axiomatic system.

Undefined terms: zork, gork, snork

Axioms:

1. For every pair of zorks z_1 and z_2, there is exactly one gork g such that z_1 snorks g and z_2 snorks g.
2. For every pair of gorks g_1 and g_2, there is a zork z that snorks both g_1 and g_2.
3. There are at least four distinct zorks, no three of which snork the same gork.

1. Let g_1 and g_2 be a pair of gorks. Fill in the blanks in the following proof that, given a pair of gorks, there is a *exactly one* zork that snorks both of them.

   Let g_1 and g_2 be a given pair of gorks. By Axiom ____, some zork z snorks both of them. Suppose another zork z' also snorks both g_1 and g_2. Then ____ and ____ are each snorked by both ____ and ____, contradicting Axiom ____. So there can't be such a zork z', and therefore there is only one zork that snorks both gorks.

2. Draw a model for this system in which a zork is a point, a gork is a line, and “snorks” means “lies on.” Use as few zorks as possible.
3. In your model, are there three gorks that are snorked by the same zork? Must this always be the case?

The next example is referred to in the exercises.

Example 1.17 Badda-Bing axiomatic system.

Undefined terms: badda, bing, hit

Axioms:

1. Every badda hits exactly four bings.
2. Every bing is hit by exactly two baddas.
3. If x and y are distinct baddas, each hitting bing q, then there are no other bings hit by both x and y.
4. There is at least one bing.

One possible model for the Badda-Bing system is shown in Figure 1.2. The picture shows an infinite collection of squares; the central square connects to four other squares whose sides are half as long. Each of these squares connects to three other smaller squares, and each of those connects to three others, and so on. This is an example of a *fractal*—a shape with some sort of infinitely repetitive geometric structure. (We'll say more about fractals in Chapter 3.)

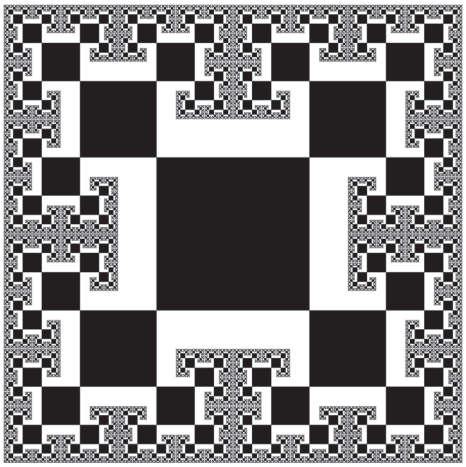


Figure 1.2 A fractal model for the Badda-Bing geometry.

In this model, a “badda” is a square, and a “bing” is a corner, or *vertex,* of a square. A square “hits” a vertex if the vertex belongs to the square. Since every square has four vertices, Axiom 1 is satisfied. Axiom 2 holds because every vertex in the model belongs to exactly two squares. Axiom 3 is a little harder to see: if squares x and y share a vertex q, there is no way they can share another vertex. And Axiom 4 is obviously true—there are lots of bings.

Exercises 1.4

1. Look up the word “root” in a dictionary. It should have several different definitions. Find a definition that is (a) descriptive and another definition that is (b) stipulative.
2. Find another word in the English language that has both descriptive and stipulative definitions.
3. Use Definition 1.5 to explain why 104 is an even integer.
4. Let n be an integer. Use Definition 1.6 to explain why $2n + 7$ is an odd integer.

5. Let n_1 and n_2 be even integers.
   (a) Use Definition 1.5 to write n_1 and n_2 in terms of integers k_1 and k_2, respectively.
   (b) Write the product $n_1 n_2$ in terms of k_1 and k_2. Simplify your answer.
   (c) Write the sum $n_1 + n_2$ in terms of k_1 and k_2. Simplify your answer.
6. Consider the following definition of the "$\lhd$" symbol.

   Definition. Let x and y be integers. Write $x \lhd y$ if $3x + 5y = 7k$ for some integer k.

   (a) Show that $1 \lhd 5$, $3 \lhd 1$, and $0 \lhd 7$.
   (b) Find a counterexample to the following statement:

   If $a \lhd b$ and $c \lhd d$, then $a \cdot c \lhd b \cdot d$.
7. Give three adjectives that describe your concept image of a circle.
8. There are several different models for geometries in which the points are ordered pairs (x, y) of real numbers; we plot these points in the usual way in the xy-plane. In such a geometry, there can be a formula for the *distance* between two points (x_1, y_1) and (x_2, y_2). For example, in Euclidean geometry, distance is given by the usual Euclidean distance formula:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

   In any geometry with a distance formula, we can define a *circle* as follows.

   Definition 1.9 A *circle* centered at (a, b) with radius r is the collection of all points (x, y) whose distance from (a, b) is r.

   (a) Use Definition 1.9 to give an equation for the circle with radius 5 centered at $(0, 0)$ in the Euclidean plane.
   (b) Plot the circle from part (a) in the xy-plane.
   (c) In the *Taxicab geometry*, the distance between two points (x_1, y_1) and (x_2, y_2) is given by the following formula.

$$\text{Distance} = |x_2 - x_1| + |y_2 - y_1|$$

   (This is called "taxicab" distance because it models the distance you would have to travel if you were restricted to driving on a rectangular city grid.) In this model, use Definition 1.9 to plot the "circle" with radius 5 centered at $(0, 0)$.
   (d) Which type of circle (Euclidean or taxicab) agrees with your concept image of circle?
9. Consider the lines $y = 2x + 1$ and $y = x + 2$ in the usual xy-plane. Use Definition 1.4 to explain why these lines are not parallel. Be specific.
10. Consider the domain of all quadrilaterals. Let

$$A(x) = \text{“}x \text{ has four right angles.”}$$
$$R(x) = \text{“}x \text{ is a rectangle.”}$$

   Write the meaning of each mathematical statement in predicate logic, keeping in mind the logical distinction between definitions and theorems.
   (a) **Definition.** A quadrilateral is a *rectangle* if it has four right angles.
   (b) **Theorem.** A quadrilateral is a rectangle if it has four right angles.
11. Write Definition 1.5 in predicate logic. Use the predicate $E(x) =$ "x is even" in the domain of integers.
12. Let the following statements be given.

   Definition. A triangle is *scalene* if all of its sides have different lengths.

   Theorem. A triangle is scalene if it is a right triangle that is not isosceles.

   Suppose $\triangle ABC$ is a scalene triangle. Which of the following conclusions are valid? Why or why not?
   (a) All of the sides of $\triangle ABC$ have different lengths.
   (b) $\triangle ABC$ is a right triangle that is not isosceles.

13. What is the difference between an axiom and a theorem?
14. Let $P(n, x, y, z)$ be the predicate "$x^n + y^n = z^n$."
    (a) Write the following statement in predicate logic, using positive integers as the domain.

    For every positive integer n, there exist positive integers x, y, and z such that $x^n + y^n = z^n$.

    (b) Formally negate your predicate logic statement from part (a). Simplify so that no quantifier lies within the scope of a negation.
    (c) In order to produce a counterexample to the statement in part (a), what, specifically, would you have to find?
15. Find a counterexample for each statement.
    (a) If n is prime, then $2^n - 1$ is prime.
    (b) Every triangle has at least one obtuse angle.[3]
    (c) For all real numbers x, $x^2 \geq x$.
    (d) For every positive nonprime integer n, if some prime p divides n, then some other prime q (with $q \neq p$) also divides n.
16. Find a counterexample for each statement.
    (a) If all the sides of a quadrilateral have equal lengths, then the diagonals of the quadrilateral have equal lengths.
    (b) For every real number $N > 0$, there is some real number x such that $Nx > x$.
    (c) Let l, m, and n be lines in the plane. If $l \perp m$ and n intersects l, then n intersects m.
    (d) If p is prime, then $p^2 + 4$ is prime.
17. Which of the statements in the previous problem can be proved as theorems?
18. Consider the following theorem.

    Theorem. Let x be a wamel. If x has been schlumpfed, then x is a borfin.

    Answer the following questions.
    (a) Give the converse of this theorem.
    (b) Give the contrapositive of this theorem.
    (c) Which statement, (a) or (b), is logically equivalent to the Theorem?
19. Draw a model for the axiomatic system of four-point geometry (Example 1.16), where a "line" is a line segment, a "point" is an endpoint of a line segment, and a point "is on" a line if it is one of its endpoints.
20. In four-point geometry, use the axioms to explain why every point is on three different lines.
21. In four-point geometry, is it possible for two different lines to both pass through two given distinct points? Explain why or why not using the axioms.
22. In four-point geometry, do triangles exist? In other words, is it possible to have three distinct points, not on the same line, such that a line passes through each pair of points? Why or why not?
23. In four-point geometry, state a good definition to stipulate what it means for two lines to *intersect*.
24. Consider the following model for four-point geometry.

    Points: 1, 2, 3, 4

    Lines: [1 2], [1 3], [1 4], [2 3], [2 4], [3 4]

    A point "is on" a line if the line's box contains the point.
    (a) Give a pair of parallel lines in this model. (Refer to Definition 1.8.)
    (b) Give a pair of intersecting lines in this model. (Use your definition from Exercise 23.)
25. Explain why, in the axiomatic system of Example 1.17, there must be at least seven distinct bings.

3. An angle is *obtuse* if it has measure greater than 90°.

26. Consider the following definition in the system of Example 1.17.

    Definition. Let x and y be distinct baddas. We say that a bing q is a *boom* of x and y, if x hits q and y hits q.

    Rewrite Axiom 3 using this definition.

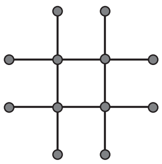
27. In the context of Example 1.17, consider the following predicates.

$$N(x,y) = \text{“}x \neq y.\text{”}$$
$$D(x) = \text{“}x \text{ is a badda.”}$$
$$G(x) = \text{“}x \text{ is a bing.”}$$
$$H(x,y) = \text{“}x \text{ hits } y.\text{”}$$

    Use these predicates to write Axiom 3 in predicate logic.

28. Refer to Example 1.17 and Figure 1.2. Describe a different model, using squares and vertices, where all the squares are the same size.

29. In the axiomatic system of Example 1.17, let a “badda” be a line segment, let a “bing” be a point, and say that a line segment “hits” a point if it passes through it. In the diagram below, there are 4 baddas and 12 bings. Is this a model for the system? Which of the axioms does this model satisfy? Explain.



30. Describe a model for Example 1.17 with 10 bings, where a “badda” is a line segment and a “bing” is a point.

1.5 Methods of Proof

The types of proofs we did in Section 1.2 were fairly mechanical. We started with the given and constructed a sequence of conclusions, each justified by a deduction rule. We were able to write proofs this way because our mathematical system, propositional logic, was fairly small. Most mathematical contexts are much more complicated; there are more definitions, more axioms, and more complex statements to analyze. These more complicated situations do not easily lend themselves to the kind of structured proof sequences of Section 1.2. In this section we will look at some of the ways proofs are done in mathematics.

Preview 1.5

1. Suppose that a and b are odd integers. What can be said about their sum $a+b$? Explain your reasoning.
2. What can be said about the sum of two even integers? Explain.

1.5.1 Direct Proofs

The structure of a proof sequence in propositional logic is straightforward: in order to prove $A \Rightarrow C$, we prove a sequence of results.

$$A \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_n \Rightarrow C$$

A *direct proof* in mathematics has the same logic, but we don’t usually write such proofs as lists of statement and reasons. Instead, this linear chain of implications is couched in mathematical prose and written in paragraph form.

Example 1.18 The proof of Theorem 1.1 on page 35 is a direct proof. Although this proof takes the form of a paragraph, the logical sequence of implications is easy to see.

There are distinct points x, y, and z.
$\Rightarrow$ There is a line l_1 through x and y, and a line l_2 through y and z.
$\Rightarrow$ x, y, and z are not on the same line, so $l_1 \neq l_2$.

These three statements are justified by Axioms 3, 1, and 4, respectively.

Example 1.19 Prove the following statement.

For all real numbers x, if $x > 1$, then $x^2 > 1$.

Proof Let x be a real number, and suppose $x > 1$. Multiplying both sides of this inequality by a positive number preserves the inequality, so we can multiply both sides by x to obtain $x^2 > x$. Since $x > 1$, we have $x^2 > x > 1$, or $x^2 > 1$, as required. □

It is worth looking back at this proof. The chain of implications is as follows.

$$x > 1 \Rightarrow x^2 > x \Rightarrow x^2 > 1 \tag{1.5.1}$$

Each conclusion is justified by an elementary fact from high school algebra, and the results are packaged in paragraph form. More precisely, the statement we were proving was actually a quantified statement of the form $(\forall x)(P(x) \to Q(x))$, where $P(x)$ means "$x > 1$" and $Q(x)$ means "$x^2 > 1$." We see that the sequence of implications in Equation (1.5.1) is true no matter what value we initially choose for x. This is the meaning of the introductory phrase "Let x be a real number." We assume nothing about x other than that it is a real number; it is arbitrary in every other respect. We then treat $P(x)$ as given and try to conclude $Q(x)$. Since x could have been any real number to start with, we have proved the implication for *all* x.

We state this type of proof as our first "Rule of Thumb" for proving theorems.

Rule of Thumb 1.1 To prove a statement of the form $(\forall x)(P(x) \to Q(x))$, begin your proof with a sentence of the form

Let x be [an element of the domain], and suppose $P(x)$.

A direct proof is then a sequence of justified conclusions culminating in $Q(x)$.

Activity 1.5.1: Direct Proofs

Recall that we define an integer n to be *even* if $n = 2k$ for some integer k. For the problems below, use the predicate $E(x)$ for "x is even."

1. Consider the statement: "For all integers n, if n is even, then n^2 is even."
   (a) Write this statement in predicate logic.
   (b) Fill in the blanks in the following proof of this statement.

   Let n be an integer, and suppose that ____________. By the definition of *even*, $n =$ _____ for some integer k. Therefore, $n^2 =$ _____ $= 2\cdot$_____, so by the definition of *even*, n^2 is even.
2. Consider the statement: "For all integers a and b, if a is even and b is even, then $a + b$ is even."
   (a) Write this statement in predicate logic.
   (b) Fill in the blanks in the following proof of this statement.

   Let a and b be integers, and suppose that ____________. By the definition of *even*, $a =$ _____ for some integer k_1 and $b =$ _____ for some integer k_2. Therefore, ... (Finish the proof.)

Before we look at another example of direct proof, we will need some tools for dealing with integers. We'll start with a definition for what it means for an integer x to *divide* another integer y.

Definition 1.10 An integer x *divides* an integer y if there is some integer k such that $y = kx$.

We write $x \mid y$ to denote that x divides y. An identical definition holds for natural numbers (i.e., positive integers). Just replace the three occurrences of "integer" in Definition 1.10 with "natural number."

We are not going to develop a rigorous axiomatic approach to the integers; such a treatment is beyond the scope of this course. When you deal with integer equations, feel free to use elementary facts from high school algebra. You can add something to both sides of an equation, use the distributive property, combine terms, and so on. However, there are certain facts about the integers that we will state as axioms, because they justify important steps in the proofs that follow.

Axiom 1.1 If a and b are integers, so are $a + b$ and $a \cdot b$.

Axiom 1.1 describes the *closure* property of the integers under addition and multiplication. Most number systems are closed under these two operations; you can't get a new kind of number by adding or multiplying. On the other hand, the integers are not closed under division: $2/3$ is not an integer, even though 2 and 3 are.

Example 1.20 Prove the following.

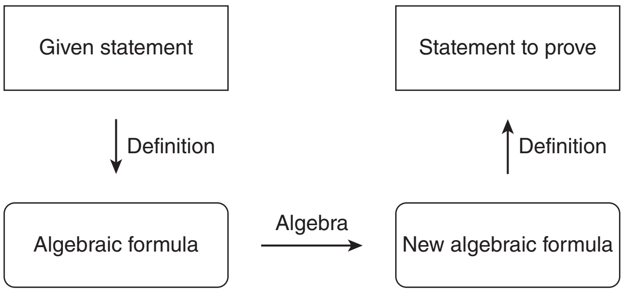
For all integers a, b, and c, if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Proof Let integers a, b, and c be given, and suppose $a \mid b$ and $a \mid c$. Then, by Definition 1.10, there is some integer k_1 such that $b = k_1 a$ and there is some integer k_2 such that $c = k_2 a$. Therefore,

$$b + c = k_1 a + k_2 a = (k_1 + k_2)a.$$

By Axiom 1.1, $k_1 + k_2$ is an integer, so $a \mid (b + c)$, again by Definition 1.10. □

Notice that this proof illustrates how definitions are used in mathematics. We used the definition of "divides" in order to translate the given statement into an equation, we did some simple algebra on this equation to obtain a new equation, and we used the definition again to translate the new equation into the statement we were trying to prove. The following "flow chart" illustrates this proof technique.



1.5.2 Proof by Contraposition

Sometimes it is hard to see how to get a direct proof started. If you get stuck (and you will), try proving the contrapositive. This is certainly permitted, since the contrapositive of a statement is its logical equivalent. We can state this as another rule of thumb.

Rule of Thumb 1.2 To prove a statement of the form $(\forall x)(P(x) \rightarrow Q(x))$, begin your proof with a sentence of the form

Let x be [an element of the domain], and suppose $\neg Q(x)$.

A proof by contraposition is then a sequence of justified conclusions culminating in $\neg P(x)$.

Example 1.21 Suppose x and y are positive real numbers such that the geometric mean $\sqrt{xy}$ is different from the arithmetic mean $\frac{x+y}{2}$. Then $x \neq y$.

Proof (By contraposition.) Let x and y be positive real numbers, and suppose $x = y$. then

$$\begin{aligned}
\sqrt{xy} &= \sqrt{x^2} && \text{since } x = y \\
&= x && \text{since } x \text{ is positive} \\
&= \frac{x+x}{2} && \text{using arithmetic} \\
&= \frac{x+y}{2} && \text{since } x = y
\end{aligned}$$

□

Contraposition isn't a radically new proof technique; a proof of a statement by contraposition is just a direct proof of the statement's contrapositive. In Example 1.21, the form of the statement to prove gave a clue that a proof by contraposition would work. If A is the statement "$\sqrt{xy} = \frac{x+y}{2}$" and B is the statement "$x = y$," then the statement to prove has the form $\neg A \to \neg B$. The contrapositive of this statement is $B \to A$, so our proof started with the assumption that $x = y$ and concluded that $\sqrt{xy} = \frac{x+y}{2}$.

For the next example we need some facts from the system of plane geometry that you studied in high school. Henceforth, we'll refer to this type of geometry as *Euclidean* geometry. The following theorem, which we will not prove, is true in Euclidean geometry.

Theorem 1.2 *The sum of the measures of the angles of any triangle equals* $180°$.

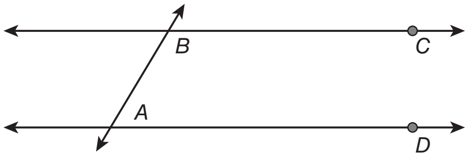
The definition of *parallel* that we used in four-point geometry also works in Euclidean geometry. Although the wording of the following definition is a little different, the content is fundamentally the same.

Definition 1.11 Two lines are parallel if they do not intersect.

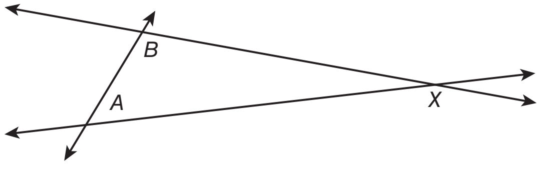
We'll use these two statements in the next example.

Example 1.22 Prove:

> If two lines are cut by a transversal such that a pair of interior angles are supplementary, then the lines are parallel.



Proof (By contraposition.) Suppose we are given two lines cut by a transversal as shown above, and suppose the lines are not parallel. Then, by the definition of parallel lines, the lines intersect. Without loss of generality, suppose they intersect on the right at point X. (If they intersect on the left, the same argument will work.)



By Theorem 1.2, the sum of the angles of $\triangle XAB$ is $180°$. Since $\angle X$ has measure greater than 0, the sum of the measures of $\angle A$ and $\angle B$ must be less than $180°$, so $\angle A$ and $\angle B$ can't be supplementary. □

1.5.3 Proof by Contradiction

Sometimes even a simple-looking statement can be hard to prove directly, with or without contraposition. In this case, it sometimes helps to try a *proof by contradiction*. The idea is a little counterintuitive. To prove statement A, suppose its negation $\neg A$ is true. Then argue, as in a direct proof, until you reach a statement that you know to be false. You will have established the sequence

$$\neg A \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_n \Rightarrow \mathbf{F}$$

where **F** represents a statement that is always false, that is, a contradiction. Taking contrapositives of this chain gives us a sequence

$$A \Leftarrow \neg B_1 \Leftarrow \neg B_2 \Leftarrow \cdots \Leftarrow \neg B_n \Leftarrow \mathbf{T}$$

and since **T** is always true (i.e., a tautology) it follows that A is true also. To sum up:

Rule of Thumb 1.3 To prove a statement A by contradiction, begin your proof with the following sentence:

> Suppose, to the contrary, that $\neg A$.

Then argue, as in a direct proof, until you reach a contradiction.

This next example is similar to Example 1.22. In fact, it is a weaker statement, so the proof given in Example 1.22 could also be used to prove it. But it makes a nice example of the contradiction method.

Example 1.23 In Euclidean geometry, prove:

> If two lines share a common perpendicular, then the lines are parallel.

Before stating the proof, notice that this theorem is of the following form.

$$(\forall x)(\forall y)(C(x,y) \rightarrow P(x,y))$$

Here $C(x,y)$ means "x and y share a common perpendicular," and $P(x,y)$ means "$x \parallel y$." You can check that the formal negation of this statement is the following.

$$(\exists x)(\exists y)(C(x,y) \wedge \neg P(x,y))$$

The translation of this last statement is "There exist lines that share a common perpendicular but are not parallel." So we use this statement to start our proof by contradiction.

Proof (By contradiction.) Suppose, to the contrary, that line AB is a common perpendicular to lines AC and BD, and also that AC and BD are not parallel. Then, by Definition 1.11, AC and BD intersect in some point X. But then $\triangle ABX$ has two right angles (and a third angle of nonzero measure), contradicting Theorem 1.2. □

The next results rely on properties of even and odd numbers, so we need to use these definitions in our arguments. Recall:

> **Definition 1.5.** An integer n is even if $n = 2k$ for some integer k.
>
> **Definition 1.6.** An integer n is odd if $n = 2k + 1$ for some integer k.

As we discussed in Section 1.4.1, these definitions alone don't imply that every integer is either even or odd. We'll state this fact as an axiom.[4]

Axiom 1.2 For all integers n, $\neg(n \text{ is even}) \Leftrightarrow (n \text{ is odd})$.

4. In a more rigorous treatment of number theory, this fact could be proved using the division algorithm, which would follow from the well-ordering principle.

In other words, any integer is either even or odd, but never both. This axiom is the key to proving the following lemma.

Lemma 1.1 Let n be an integer. If n^2 is even, then n is even.

The following activity will lead you through two different proofs of this lemma.

Activity 1.5.2: Proofs by Contraposition and Contradiction

Consider statement P: "For all integers n, if n^2 is even, then n is even."

1. Write statement P in predicate logic. Use $E(x)$ for "x is even."
2. Write the contrapositive of statement P in predicate logic. Use $O(x)$ for "x is odd."
3. Give a direct proof of the contrapositive of statement P.
4. Write the negation of P in predicate logic. Use rules of logic to write this statement using the $\exists$ quantifier and the $\wedge$ connective, and translate this negation back into English.
5. Assume that the negation of P is true. Prove that this assumption leads to a contradiction.

Our final example is a classic proof by contradiction. Recall that a *rational* number is a number that can be written as a/b, where a and b are integers with $b \neq 0$.

Example 1.24 Prove that $\sqrt{2}$ is irrational.

Proof (By contradiction.) Suppose, to the contrary, that $\sqrt{2}$ is rational, so there are integers a and b such that $a/b = \sqrt{2}$, and a and b can be chosen so that the fraction a/b is in lowest terms. Then $a^2/b^2 = 2$, so $a^2 = 2b^2$, that is, a^2 is even. By Lemma 1.1, a is even. Therefore, $a = 2k$ for some integer k, so $a^2 = 4k^2$. But now we have $b^2 = a^2/2 = 2k^2$, so b^2 is even, and therefore, by the lemma again, b is even as well. We have shown that a and b are both even, which contradicts the assumption that a/b is in lowest terms. □

Exercises 1.5

1. Consider the following statement.

   For all integers x, if $4 \mid x$, then x is even.

   (a) Write this statement in predicate logic in the domain of integers. Say what your predicates are.
   (b) Apply Rule of Thumb 1.1 to write down the first sentence of a direct proof of this statement.
   (c) Use Definition 1.10 to translate your supposition in part (b) into algebra.
   (d) Finish the proof of the statement.
2. Give a direct proof:

   Let a, b, and c be integers. If $a \mid b$ and $a \mid c$, then $a \mid (b \cdot c)$.

   Remember that you must use the definition of | in your proof.
3. Prove: Let a, b, and c be integers. If $(a \cdot b) \mid c$, then $a \mid c$.
4. Give a direct proof.

   Let a, b, and c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.
5. Give a direct proof of the following statement in Euclidean geometry. Cite any theorems you use.

   The sum of the measures of the angles of a parallelogram is 360°.

6. Prove:

   For all integers n, if n^2 is odd, then n is odd.

   Use a proof by contraposition, as in Lemma 1.1.
7. Prove the following statement by contraposition.

   Let x be an integer. If $x^2 + x + 1$ is even, then x is odd.

   Make sure that your proof makes appropriate use of Definitions 1.5 and 1.6.
8. Prove that the sum of two even integers is even.
9. Prove that the sum of an even integer and an odd integer is odd.
10. Prove that the sum of two odd integers is even.
11. Write a proof by contradiction of the following.

    Let x and y be integers. If x and y satisfy the equation

    $$3x + 5y = 153$$

    then at least one of x and y is odd.
12. Prove the following statement in Euclidean geometry. Use a proof by contradiction.

    A triangle cannot have more than one obtuse angle.
13. Let "$x \nmid y$" denote "x does not divide y." Prove by any method.

    Let a and b be integers. If $5 \nmid ab$, then $5 \nmid a$ and $5 \nmid b$.
14. Consider the following definition.

    Definition. An integer n is *sane* if $3 \mid (n^2 + 2n)$.

    (a) Give a counterexample to the following: All odd integers are sane.
    (b) Give a direct proof of the following: If $3 \mid n$, then n is sane.
    (c) Prove by contradiction: If $n = 3j + 2$ for some integer j, then n is not sane.
15. Recall Exercise 6 of Section 1.4. Consider the following definition of the "$\lhd$" symbol.

    Definition. Let x and y be integers. Write $x \lhd y$ if $3x + 5y = 7k$ for some integer k.

    Give a direct proof of the following statement.

    If $a \lhd b$ and $c \lhd d$, then $a + c \ \lhd\ b + d$.
16. Consider the following definitions.

    Definition. An integer n is *alphic* if $n = 4k + 1$ for some integer k.
    Definition. An integer n is *gammic* if $n = 4k + 3$ for some integer k.

    (a) Show that 19 is gammic.
    (b) Suppose that x is alphic and y is gammic. Prove that $x + y$ is even.
    (c) Prove by contraposition: If x is not odd, then x is not alphic.
17. Prove that the rational numbers are closed under multiplication. That is, prove that, if a and b are rational numbers, then $a \cdot b$ is a rational number.
18. Prove that the rational numbers are closed under addition.
19. Prove: Let x and y be real numbers with $x \neq 0$. If x is rational and y is irrational, then $x \cdot y$ is irrational.
20. Prove: Let x and y be real numbers. If x is rational and y is irrational, then $x + y$ is irrational.

21. Consider the following definition.

    Definition. An integer n is *frumpable* if $n^2 + 2n$ is odd.

    Prove: All frumpable numbers are odd.

22. Recall the Badda-Bing axiomatic system of Example 1.17. Prove:

    If q and r are distinct bings, both of which are hit by baddas x and y, then $x = y$.

23. Two common axioms for geometry are as follows. The undefined terms are "point," "line," and "is on."
    1. For every pair of points x and y, there is a unique line such that x is on l and y is on l.
    2. Given a line l and a point x that is not on l, there is a unique line m such that x is on m and no point on l is also on m.

    Recall that two lines l and m are *parallel* if there is no point on both l and m. In this case we write $l \parallel m$. Use this definition along with the above two axioms to prove the following.

    Let l, m, and n be distinct lines. If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.

24. Recall Example 1.16. In the axiomatic system for four-point geometry, prove the following assertion using a proof by contradiction:

    Suppose that a and b are distinct points on line u. Let v be a line such that $u \neq v$. Then a is not on v or b is not on v.

25. The following axioms characterize *projective geometry*. The undefined terms are "point," "line," and "is on."
    1. For every pair of distinct points x and y, there is a unique line l such that x is on l and y is on l.
    2. For every pair of lines l and m, there is a point x on both l and m.
    3. There are (at least) four distinct points, no three of which are on the same line.

    Prove the following statements in projective geometry.

    (a) There are no parallel lines.

    (b) For every pair of lines l and m, there is exactly one point x on both l and m.

    (c) There are (at least) four distinct lines such that no point is on three of them.