Chapter 6

Planar Symmetries

6.1 Introduction

The use of symmetry as an aspect of art goes back several millennia, and some early samples of artistic creations into which symmetry was consciously incorporated are displayed in Figure 6.1. The Greek mathematician Euclid (circa 300 BC), whose opus *The Elements* codified geometry for the following 2000 years, incorporated into this series of 13 books many propositions regarding both two- and three-dimensional symmetric figures. Interestingly, the first proposition of the first book concerns the construction of equilateral triangles and the last book is devoted to the description of the five regular solids (Fig. 7.1), all highly symmetric geometric objects. This endowed Euclid's work with a symmetry of its own. Given that he could have developed the subject matter in many other ways, it is arguable that this symmetric format was a result of a conscious decision on his part.

Euclid and his successors viewed symmetries as merely aesthetically pleasing aspects of geometric objects rather than the subjects of mathematical investigations. In other words, the symmetries of these figures attracted their attention, but their technical investigations were restricted to the mathematical consequences of these symmetries rather than the symmetries themselves. It was not until the 18th century that mathematicians realized that symmetries per se could and should be subject to mathematical research. Since then, the mathematical study of symmetry, better known as *group theory*, has evolved into a deep and essential branch of mathematics with important applications to the physical sciences. This chapter is devoted to the description of the symmetries of plane figures.

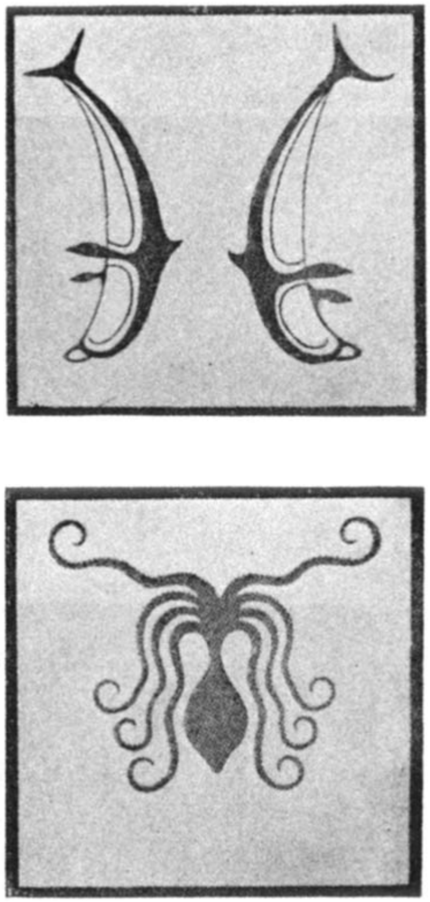


Figure 6.1 *Symmetry in ancient art*

6.2 Rigid Motions of the Plane

The *Oxford English Dictionary*'s definition of geometric symmetry is as follows:

> *Exact correspondence in position of the several points or parts of a figure or body with reference to a dividing line, plane, or a point.*

The mathematical understanding is different: A geometric symmetry is a potential transformation of an object that would leave it in the same position. This requires a fair amount of clarification, and we begin with the notion of transformation or, more technically, a rigid motion.

A rigid motion of the plane takes place when the plane is moved through space, without distortions, so that the plane as a whole ends up occupying its initial position, even though individual points and figures are deposited in new locations. Although the actual movement through space is a useful tool for visualizing a rigid motion, it is its final position that is of interest to us here, and consequently any two rigid motions that leave the points and figures of the plane in the same final positions are considered as identical, even though in their intermediary stages they may move the same points along different paths through space. Such is the case with the clockwise 90° and counterclockwise 270° turns: Both leave us facing to the right relative to our original position. In general, motions are visualized in the most efficient, or economical, manner possible. Moreover, rather than describe them as an actual movement through space, a difficult task within the confines of a textbook, the rigid motions are described by means of their final effect on some figures.

The first type of rigid motions to be described are the *translations*. These are the rigid motions that keep all line segments parallel to their initial positions. The horizontal 3-inch shift to the right is a translation that carries the collection *A* of Figure 6.2 to location *B*. Of course, the horizontal 3-inch shift to the left is the translation that carries *B* to *A*. Similarly, there is a north-easterly translation that carries the mushroom M of Figure 6.3 to *N*.

Before going on to discuss other types of rigid motions, it is necessary to make some more general definitions. If a rigid motion picks up a figure *F* and moves it to a new position G, it is said that G is the *image* of *F* under the action of that motion. Thus, *B* in Figure 6.2 is the image of A under the horizontal 3-inch rightward shift, whereas *A* is the image of *B* under the horizontal 3-inch leftward shift.

Given a figure *F* and a rigid motion *m*, the collection of all the images generated from *F* by the iteration of both that motion and its reversal constitute the *orbit* of *F* generated by *m*. *F* in Figure 6.4 depicts an orbit generated by a horizontal translation.

Figure 6.2 *A figure with its image*

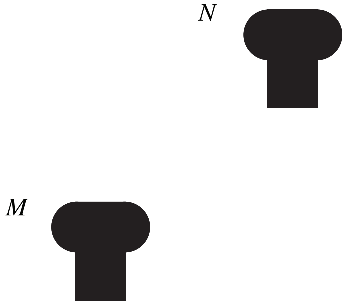


Figure 6.3 *Another figure with its image*

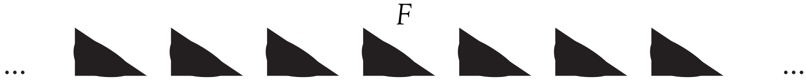


Figure 6.4 *A translation*

Figure 6.5 depicts an orbit generated by a north-easterly translation. The orbits generated by any rigid motion and its reversal are, of course, identical, provided that they begin with the same figures.

The second class of rigid motions are the *rotations*. These motions turn the plane through a certain signed angle about some pivot point (Fig. 6.6). When the specified angle is positive, the rotation moves counterclockwise from the point of view of the reader; when the angle is negative, the motion is clockwise. Although in general any angle can serve as an angle of rotation, the subsequent discussion is restricted to rotations whose angles measure 60°, 90°, 120°, and 180°. Typical orbits of such rotations appear in Figure 6.7. Their orbits are said to be 6-fold, 4-fold, 3-fold, and 2-fold, respectively. Because we are only interested in the question of where figures land

Figure 6.5 *A translation*

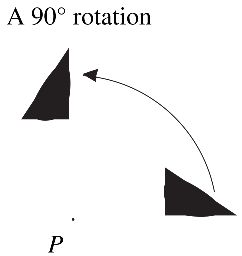


Figure 6.6 *Rotations about a pivot point* P

rather than how they got there, rotations by angles of 180° and –180° with the same pivot points are equal to each other.

Yet another class of rigid motions are the *reflections*, so called because their effect on geometric figures is akin to that of mirror reflections. They can be visualized as the effect of a 180° spatial rotation about a straight line, the *axis*, that lies in the rotated plane. The orbit of any figure generated by a reflection consists of two copies of that figure (Fig. 6.8). Because only the final position of the moving figure is of concern here, the fact that this rotation switches the "sides" of the plane is immaterial.

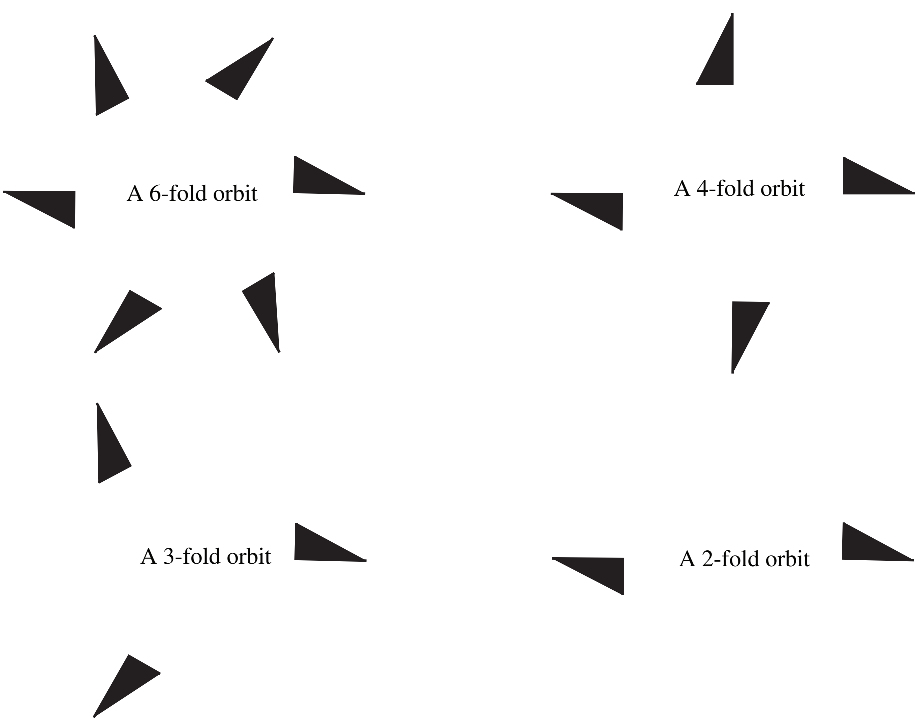


Figure 6.7 *Orbits of rotations*

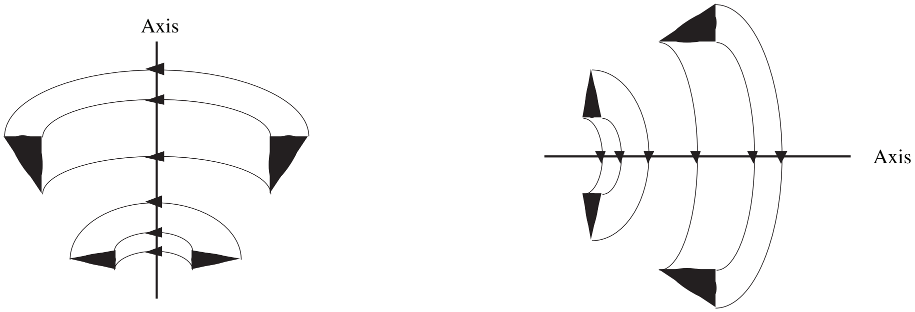


Figure 6.8 *Orbits of reflections*

The last type of rigid motion is the *glide reflection* (Fig. 6.9). As its name implies, this is a combination of a translation and a reflection whose axis is parallel to the direction of the translation. The orbit of a figure generated by a glide reflection consists of an infinite number of copies of the figure that are situated alternately on the two sides of the axis. As was the case for translations, the orbit of a figure generated by a glide reflection also includes the copies obtained by applying the reversal of the glide reflection, and so this orbit extends infinitely far in two directions (Fig. 6.10). The glide reflections complete the roster of all the rigid motions of the plane. Surprisingly, no new rigid motions are created by combining other rigid motions. This is not easy to visualize, but the combination of a translation with a rotation yields another rotation, and the combination of a rotation with a reflection generally yields a glide reflection. The following theorem is due to M. Chasles (1793–1880).

THEOREM 6.2.1. *The rigid motions of the plane consist of translations, rotations, reflections, and glide reflections.*

Just as 0 turns out to be a convenient number, so is it useful to consider the "motion" that does not move anything as a motion that is called the *identity*. Because

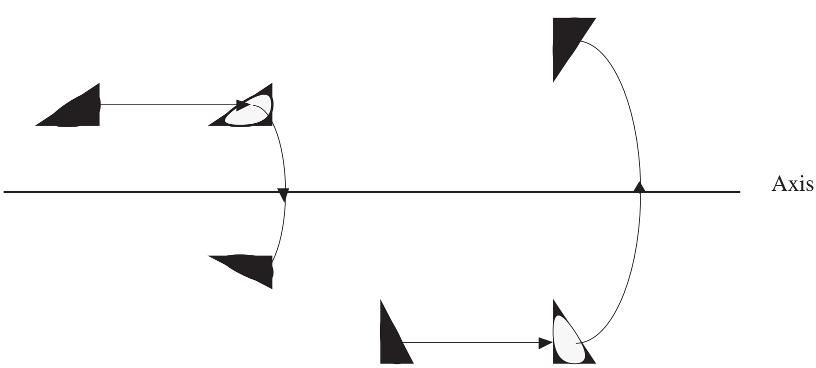


Figure 6.9 *A glide reflection*

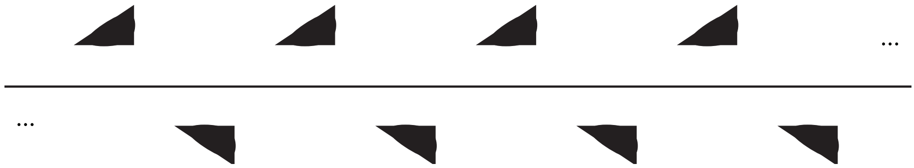


Figure 6.10 *The orbit of a glide reflection*

this motion does not change the distances between any two points, it qualifies as a rigid motion. The identity motion is denoted by *Id*.

EXERCISES 6.2

Exercises 1–13 all refer to Figure 6.11.

1. Sketch six figures in the orbit of ΔABC generated by a 1-inch translation in a direction parallel to the straight line m.
2. Sketch six figures in the orbit of ΔABC generated by a 1-inch translation in a direction parallel to the straight line n.
3. Sketch the orbit of ΔABC generated by a rotation that is pivoted at A and has angle

   a) 60° c) 120°

   b) 90° d) 180°.

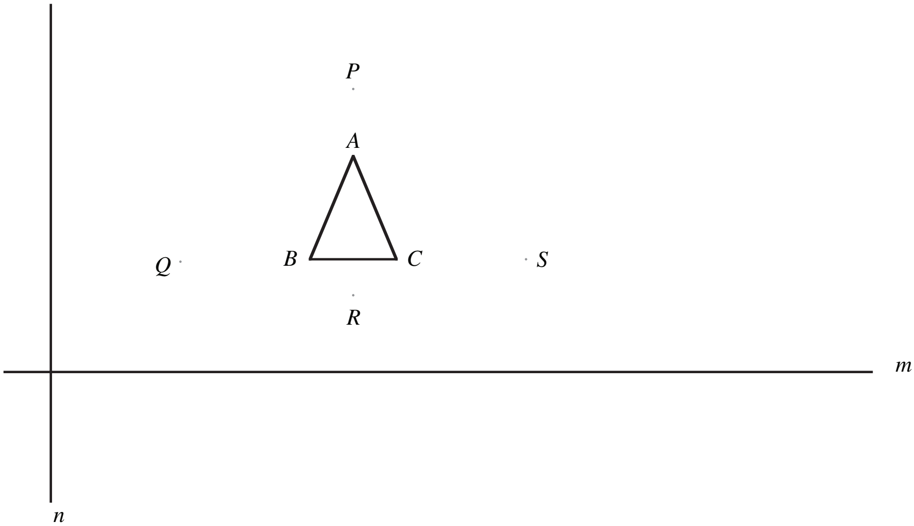


Figure 6.11 *Exercises 1–13*

4. Sketch the orbit of Δ *ABC* generated by a rotation that is pivoted at *B* and has angle
   a) 60°
   b) 90°
   c) 120°
   d) 180°.
5. Sketch the orbit of Δ *ABC* generated by a rotation that is pivoted at *P* and has angle
   a) 60°
   b) 90°
   c) 120°
   d) 180°.
6. Sketch the orbit of Δ *ABC* generated by a rotation that is pivoted at *Q* and has angle
   a) 60°
   b) 90°
   c) 120°
   d) 180°.
7. Sketch the orbit of Δ *ABC* generated by a rotation that is pivoted at *R* and has angle
   a) 60°
   b) 90°
   c) 120°
   d) 180°.
8. Sketch the orbit of Δ *ABC* generated by the reflection with axis *m*.
9. Sketch the orbit of Δ *ABC* generated by the reflection with axis *n*.
10. Sketch the orbit of Δ *ABC* generated by the reflection with axis *AB*.
11. Sketch the orbit of Δ *ABC* generated by the reflection with axis *BC*.
12. Sketch the orbit of Δ *ABC* generated by the reflection with axis *QR*.
13. Sketch the orbit of Δ *ABC* generated by the reflection with axis *RS*.
14. Sketch six figures in the orbit of Δ *ABC* generated by a 1-inch glide reflection with axis *m*.

15. Sketch six figures in the orbit of $\Delta\, ABC$ generated by a 1-inch glide reflection with axis n.

16. Explain why the 360° rotation about a point P equals the identity motion.

6.3 Symmetries of Polygons

A (*mathematical*) *symmetry* of a figure is a rigid motion that carries that figure onto itself. Alternately, a rigid motion is a symmetry of a figure provided this figure constitutes the entire orbit generated by the motion. Thus, the reflections with axes d, m, e, and n are all symmetries of the square (Fig. 6.12), and they are denoted by ρ_d, ρ_m, ρ_e, and ρ_n respectively. These reflections do not constitute all the square's symmetries. The 90°, 180°, and 270° (or –90°) rotations about the center of the square are also such symmetries. If C denotes this center, then $R_{C,90°}$, $R_{C,180°}$, $R_{C,270°}$ denote these respective rotations. The identity rigid motion *Id* is another such symmetry. The set of all the symmetries of a figure is called its *symmetry group*, or just *group*. Thus, the symmetry group of the square is

$$\{\mathrm{Id}, \rho_d, \rho_e, \rho_m, \rho_n, R_{C,90°}, R_{C,180°}, R_{C,270°}\}$$

By definition, every plane figure F has a symmetry group that contains at least the identity motion *Id*. The isosceles triangle in Figure 6.13 has $\{Id, \rho_v\}$ as its symmetry group, whereas that of the equilateral triangle is $\{Id, \rho_d, \rho_e, \rho_f, R_{C,120°}, R_{C,240°}\}$.

Because the orbits of translations and glide reflections are infinite in extent, it follows that no finite figure can have symmetries of either of these types. Infinitely extended figures, however, can have such symmetries. We first show how to create such figures and then examine the question of what their symmetry groups are like.

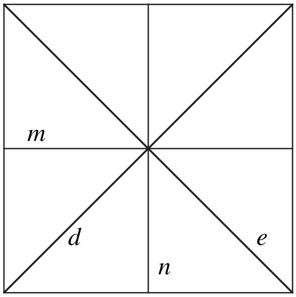


Figure 6.12 *Some symmetries of the square*

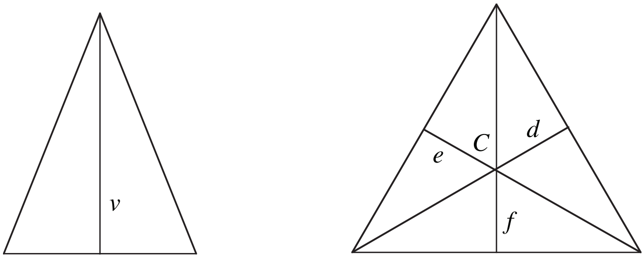


Figure 6.13 *An isosceles and an equilateral triangle*

EXERCISES 6.3

1. Write down the symmetry groups of the following figures.
   a) The rectangle with unequal sides
   b) The regular pentagon
   c) The regular hexagon
   d) The regular heptagon
   e) The regular octagon

6.4 Frieze Patterns

Orbits of finite figures generated by translations are called *frieze patterns*. These frieze patterns are the mathematical idealization of such decorative designs as borders used to accent wallpapers and trim sewn or printed around a cloth (Fig. 6.14). Unlike their physical manifestations, frieze patterns are understood to extend indefinitely in both directions, just like a straight line.

The frieze pattern of a figure F necessarily has its generating translation as a symmetry, and in addition it inherits all the symmetries of F. This observation, however, does not account for *all* the symmetries of the frieze pattern. As shown in Figure 6.15, in the case of the figure F_1, there are no other symmetries, and so the pattern's symmetry group is denoted by $\Gamma_1 = < \tau >$, where τ denotes the horizontal translation that generates the infinite frieze pattern $\Pi(F_1)$. This is not meant to imply that τ is the only symmetry of $\Pi(F_1)$. It is clear that the translation 2τ that denotes two consecutive applications of τ is also a symmetry of $\Pi(F_1)$, and similarly, for each positive integer k, the translation $k\tau$, which consists of k iterations of τ, is also a symmetry of $\Pi(F_1)$. Frieze patterns always have infinite symmetry groups, and instead of listing their elements it is customary simply to list the types of symmetries they contain.

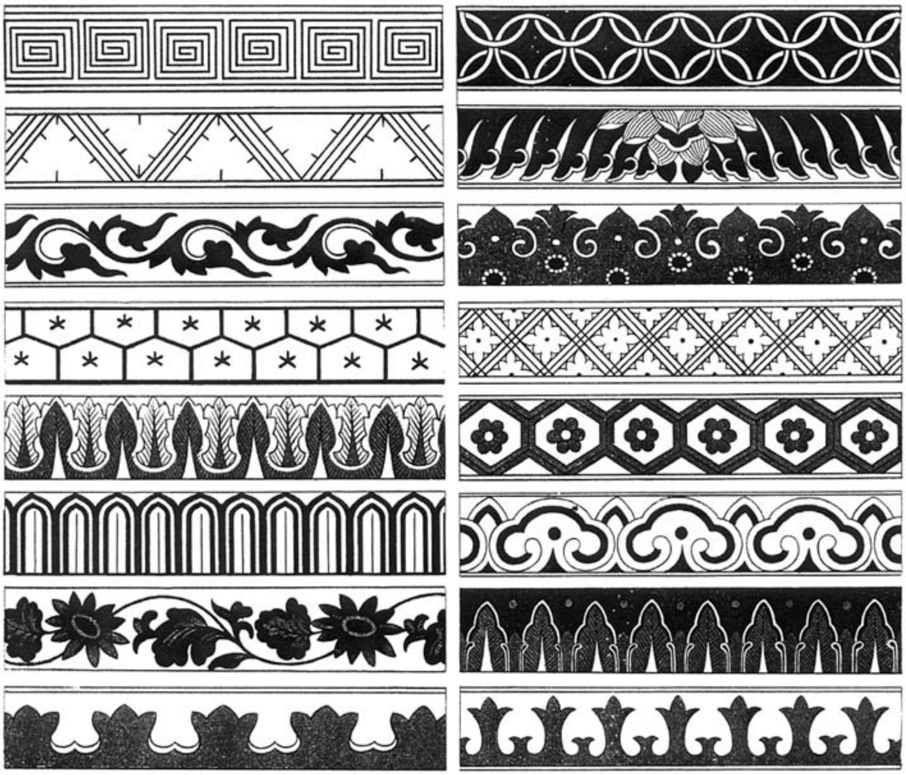


Figure 6.14 *Chinese frieze patterns*

Figure F_2 possesses the symmetry ρ_h (h for horizontal), which is of course also a symmetry of its frieze pattern $\Pi(F_2)$. In addition, this pattern also necessarily possesses the glide reflection γ obtained by combining t with ρ_h as a symmetry. This

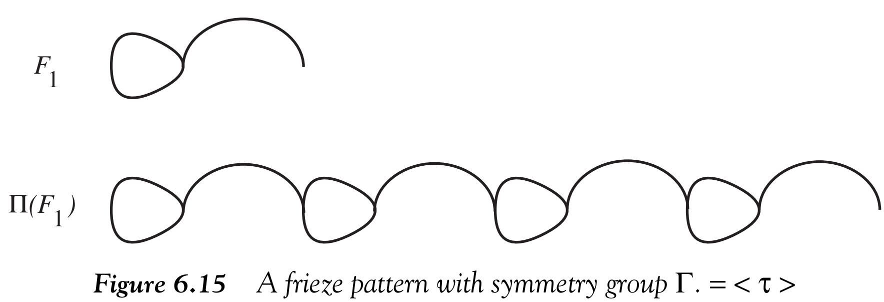


Figure 6.15 *A frieze pattern with symmetry group* $\Gamma. = < \tau >$

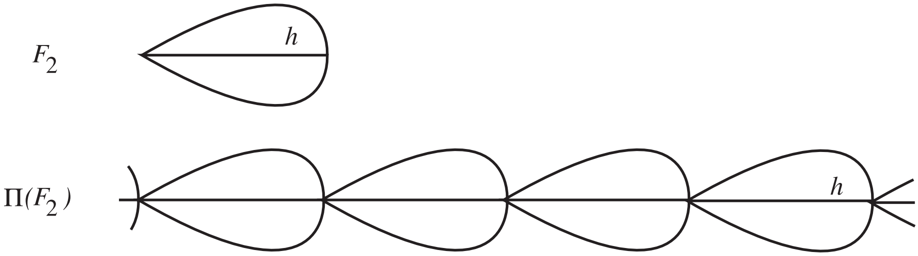


Figure 6.16 *A frieze pattern with symmetry group* $\Gamma_2 = <\tau,\ \rho_h,\ \gamma>$

frieze's symmetry group is denoted by $\Gamma_2 = <\tau,\ \rho_h,\ \gamma>$ (Fig. 6.16). The symmetry ρ_v of F_3 results in a multitude of symmetries of the frieze $\Pi(F_3)$, which are all essentially identical (Fig. 6.17). However, this frieze pattern possesses an additional symmetry, namely the reflection $\rho_{v'}$, which has no counterpart in the generating figure F_3. Because of its similarity to ρ_v, the symmetry $\rho_{v'}$ is not listed in the symmetry group $\Gamma_3 = <\tau,\ \rho_v>$ of this frieze pattern. Such an additional reflection could not have appeared with the horizontal reflection of F_2, but similar "accidental" symmetries can arise in other cases, as seen below. The symmetry $R_{C,180°}$ of the figure F_4 results in the symmetries $R = R_{C,180°}$ of the frieze pattern. Once again, the frieze pattern has the additional symmetry $R_{C',180°}$. This frieze pattern's symmetry group is $\Gamma_1 = <\tau,\ R>$ (Fig. 6.18). The figure F_5 possesses all three of the above symmetries, as does the generated frieze pattern. Its symmetry group is $\Gamma_5 = <\tau,\ \rho_h,\ \rho_v,\ R,\ \gamma>$ (Fig. 6.19). This pattern of course also has the additional symmetries described for Γ_3 and Γ_4. The patterns shown in Figures 6.20 and 6.21 have symmetry groups $\Gamma_6 = <\tau,\ \gamma>$ and $\Gamma_7 = <\tau,\ \gamma,\ R>$ that contain glide reflections, just like Γ_2. Unlike the glide reflection of Γ_2, those of Γ_6 and Γ_7 do not have their component translation and reflection in the group.

The following theorem, proved by P. Niggli in 1926, states that these are all the possible symmetry groups that frieze patterns can possess.

THEOREM 6.4.1 *Every frieze pattern has a symmetry group that is identical with one of the groups* $\Gamma_1 = <\tau>$, $\Gamma_2 = <\tau,\ \rho_h,\ \gamma>$, $\Gamma_3 = <\tau,\ \rho_v>$, $\Gamma_4 = <\tau,\ R>$, $\Gamma_5 = <\tau,\ \rho_h,\ \rho_v,\ R,\ \gamma>$, $\Gamma_6 = <\tau,\ \gamma>$, $\Gamma_7 = <\tau,\ \gamma,\ R>$.

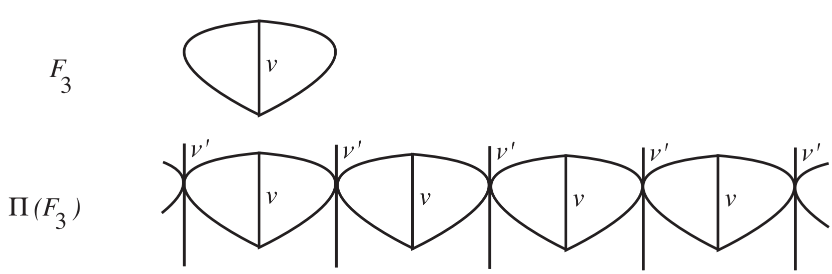


Figure 6.17 *A frieze pattern with symmetry group* $\Gamma_3 = <\tau,\ \rho_v>$

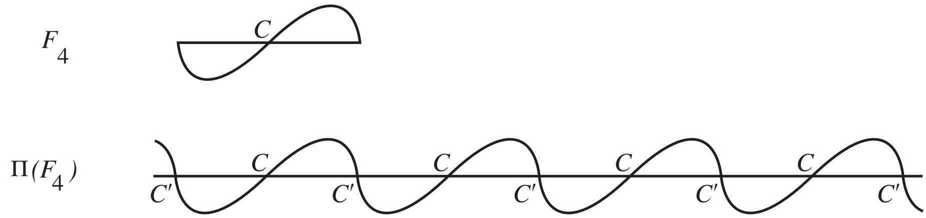


Figure 6.18 *A frieze pattern with symmetry group* $\Gamma_4 = < \tau, R >$

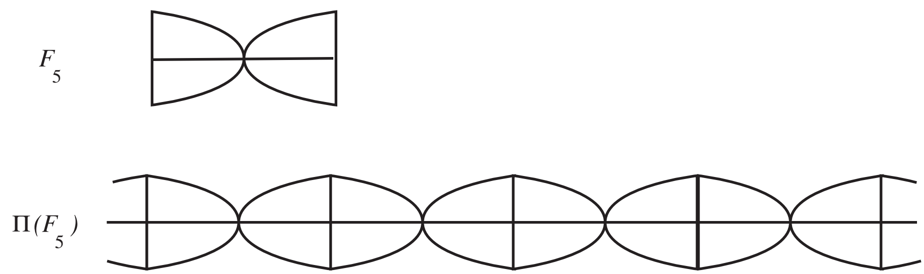


Figure 6.19 *A frieze pattern with symmetry group* $\Gamma_5 = < \tau, \rho_h, \rho_v, R, \gamma >$

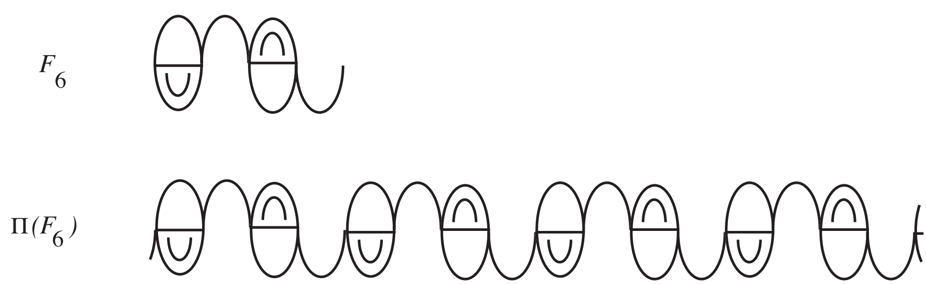


Figure 6.20 *A frieze pattern with symmetry group* $\Gamma_6 = < \tau, \gamma >$

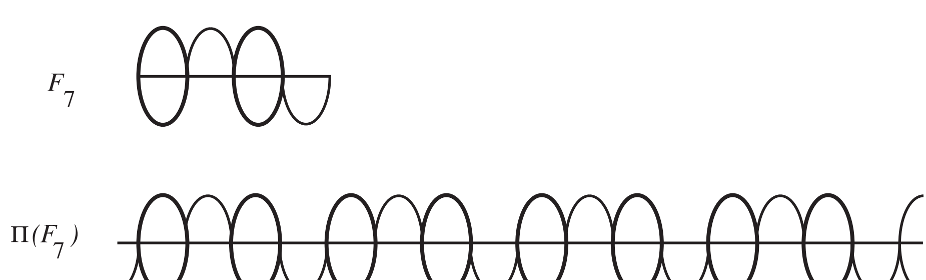
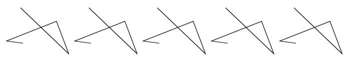
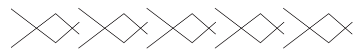
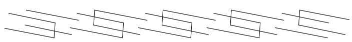
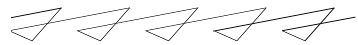
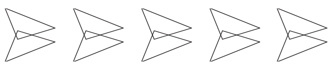
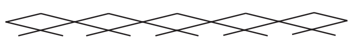
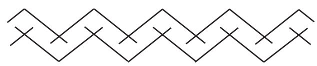
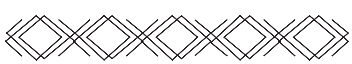
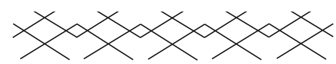
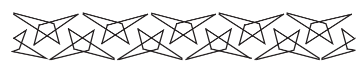
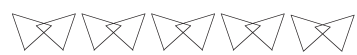
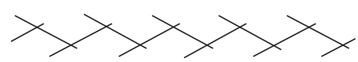
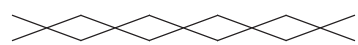
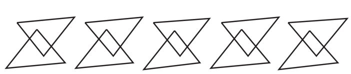
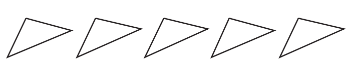
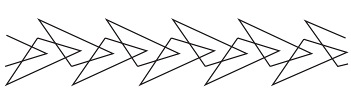


Figure 6.21 *A frieze pattern with symmetry group* $\Gamma_7 = < \tau, \gamma, R >$

EXERCISE 6.4

Identify the groups of the following frieze patterns.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. 
15. 
16. 
17. 
18. 
19. 
20. 
21. 

6.5 Wallpaper Designs

Wallpaper designs are the two-dimensional analogs of frieze patterns. More precisely, let $\Pi(F)$ denote the frieze pattern generated by a figure F and a translation τ. If τ^* is another translation whose direction is not parallel to that of τ, then the orbit $\Omega(F) = \Pi(\Pi(F))$ of τ^* generated by $\Pi(F)$ is a *wallpaper design*. In other words, a wallpaper design is an orbit of an orbit.

It is clear that both τ and τ^* are symmetries of the wallpaper design they generate, and an illustration of this appears in Figures 6.22 and 6.23. As was the case for frieze patterns, the generated design $\Pi(F)$ may possess further symmetries that are not present in F. In contrast with the seven different groups of symmetries of frieze patterns, there are 17 different possibilities for the symmetry groups of wallpaper designs. Wallpaper designs exemplifying all these are exhibited in Figures 6.24 through

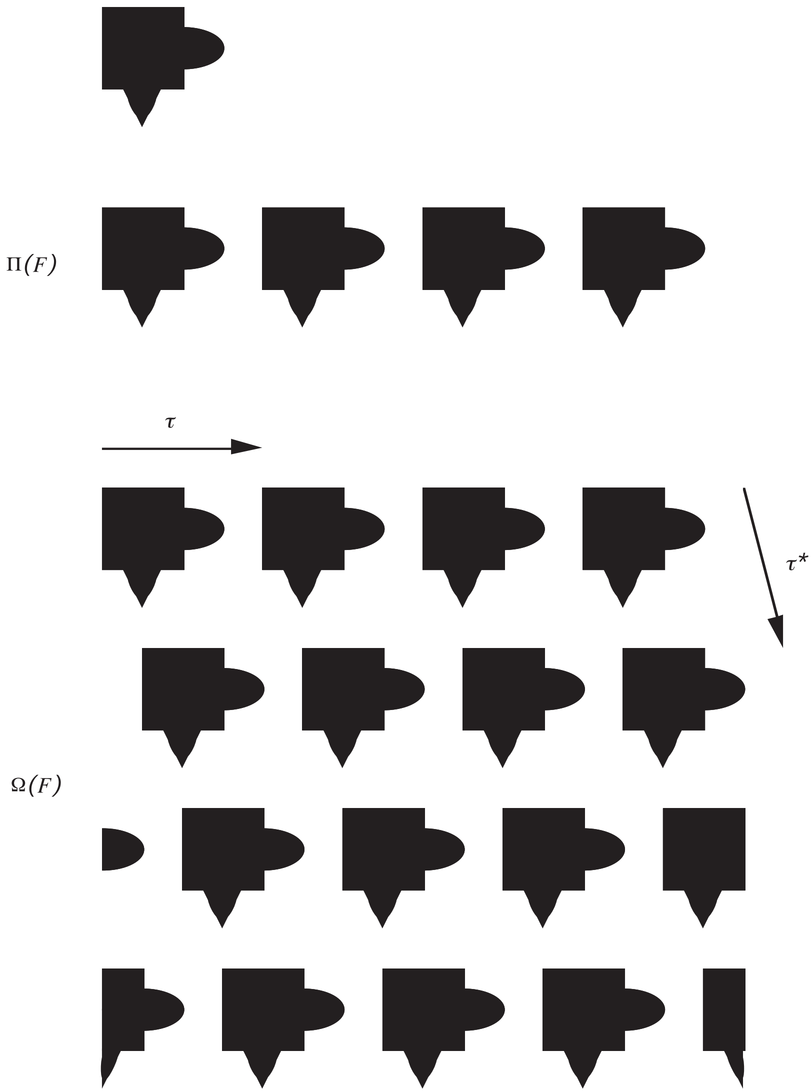


Figure 6.22 *A wallpaper design*

6.26. In these diagrams, the presence of reflections and glide reflections is denoted by a dashed line with a label of either ρ (for reflection) or γ (for glide translation). The centers of rotational symmetries are denoted by the symbols ◊ (180°), △ (120°), □ (90°), ⬡ (60°). Table 6.1 lists the salient symmetry characteristics of each design. A rotation through an angle of 360°/n is said to have *order n*. A glide reflection is said to be *nontrivial* if its component translation and reflection are not symmetries of the pattern.

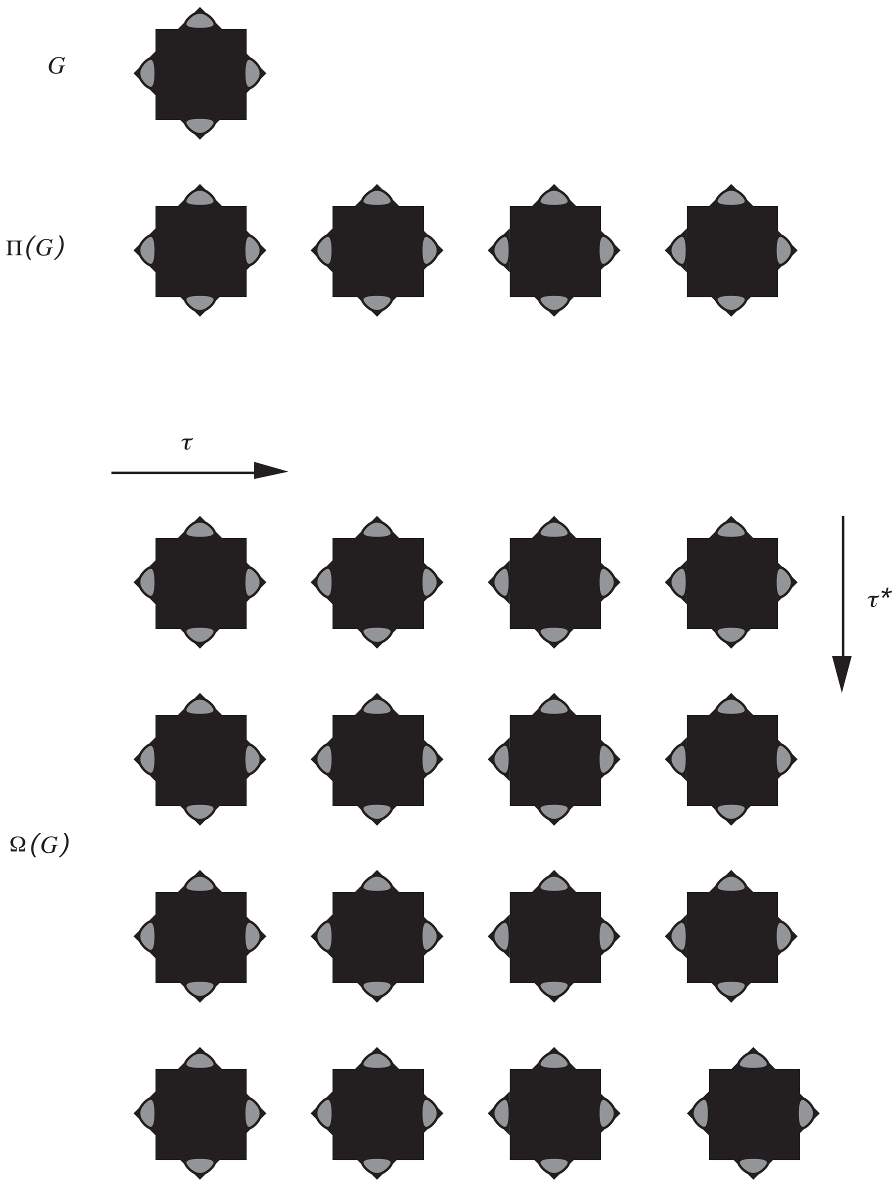


Figure 6.23 *Another wallpaper design*

The symbols *p1*, *pgg*, *p31m*, and so on listed in Table 6.1 under "Type" are used to denote both a type of wallpaper design and its symmetry group. They are known as the crystallographic notation for the symmetry groups. If the second character in this symbol is an integer, it is the highest order of all the rotations in that group. The significance of the other characters is too technical to explain here.

THEOREM 6.5.1 *There are exactly 17 wallpaper groups.*

The characteristics of the designs that correspond to these groups are displayed in Table 6.1. Figures 6.24–6.26 each display one design for each of these groups.

Theorem 6.5.1 was first discovered in 1891 by E. S. Fedorov, 35 years before Niggli stated and proved its one-dimensional analog on frieze groups (Theorem 9.4.1). Curiously, this work had been preceded by Fedorov's and Arthur Schönflies's (1853–1928) independent classifications of the 230 crystallographic groups, these being the three-dimensional analogs of the wallpaper groups. It has since been established that there are exactly 4783 classes of such groups in four-dimensional space. For spaces of more than four dimensions, it is only known that the number of such symmetry groups is finite.

Type	Highest order of rotation	Reflections	Nontrivial glide reflections	Helpful distinguishing properties
p1	1	No	No	
p2	2	No	No	
pm	1	Yes	No	
pg	1	No	Yes	
cm	1	Yes	Yes	
pmm	2	Yes	No	
pmg	2	Yes	Yes	Parallel reflection axes
pgg	2	No	Yes	
cmm	2	Yes	Yes	Perpendicular reflection axes
p4	4	No	No	
p4m	4	Yes	Yes	4-fold centers on reflection axes
p4g	4	Yes	Yes	4-fold centers not on reflection axes
p3	3	No	No	
p3m1	3	Yes	Yes	All 3-fold centers on reflection axes
p31m	3	Yes	Yes	Not all 3-fold centers on reflection axes
p6	6	No	No	
p6m	6	Yes	Yes	

From Doris Schattschneider's article.

Table 6.1 Recognition chart for plane periodic patterns

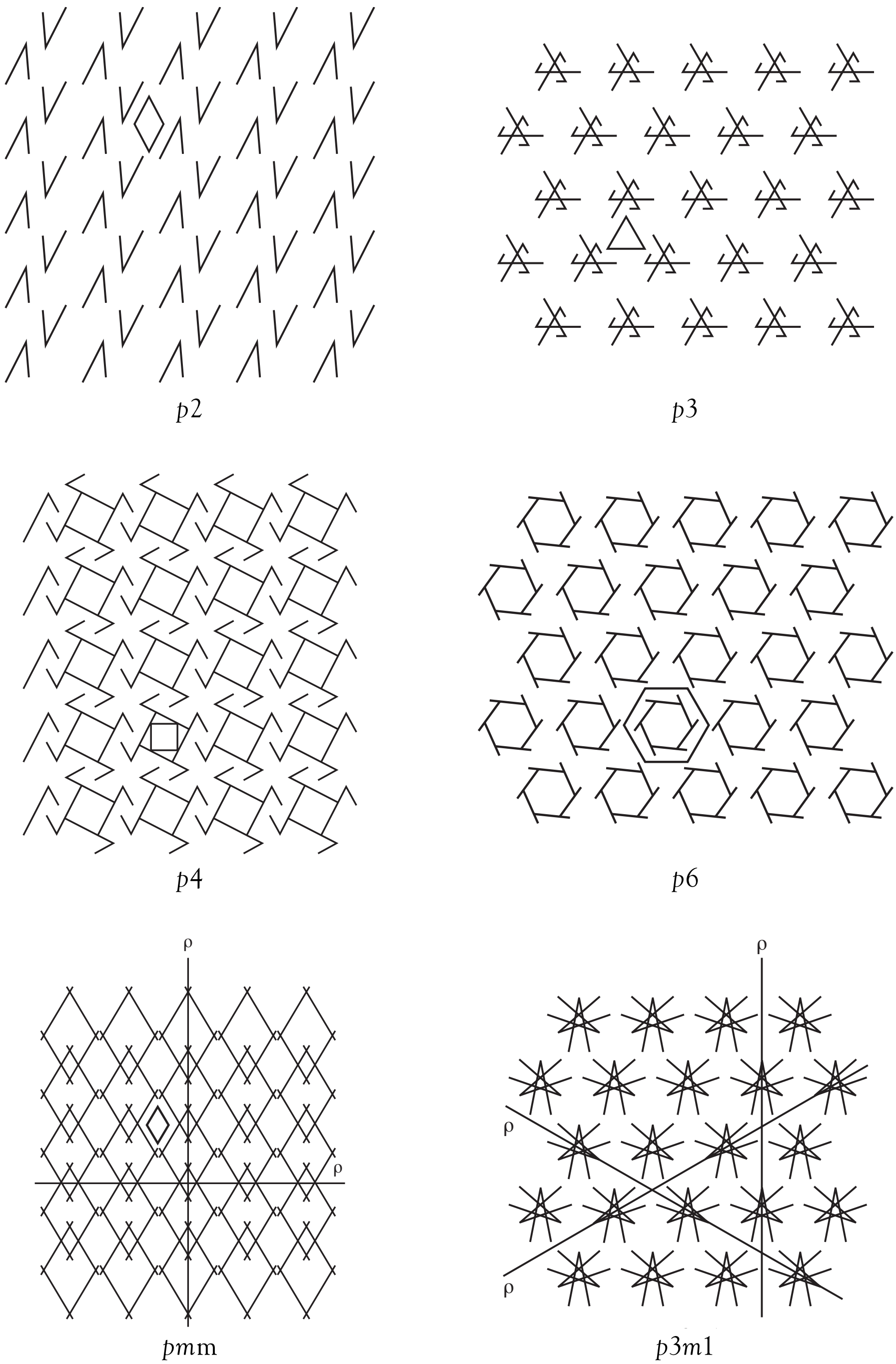


Figure 6.24 *Six wallpaper designs*

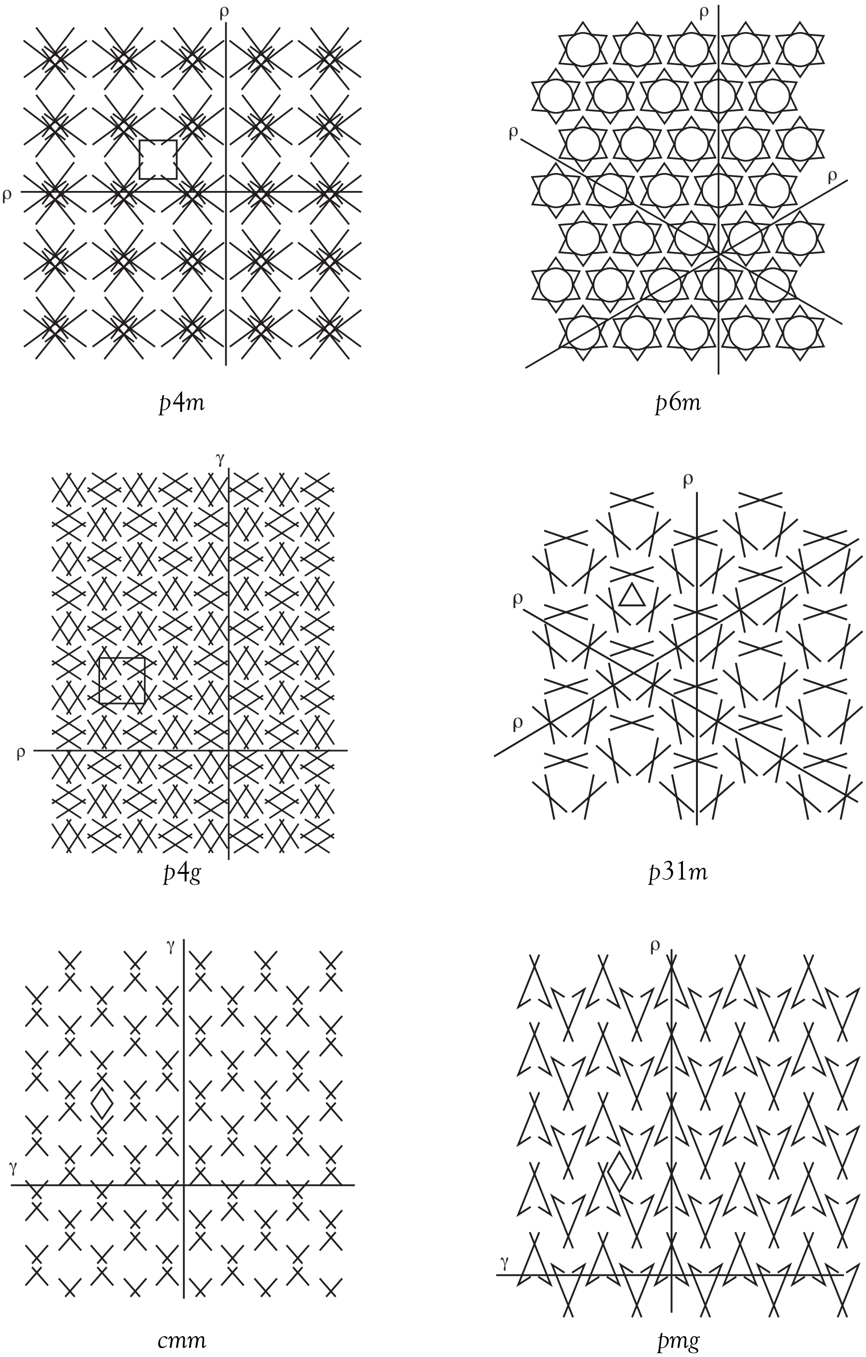


Figure 6.25 *Six more wallpaper designs*

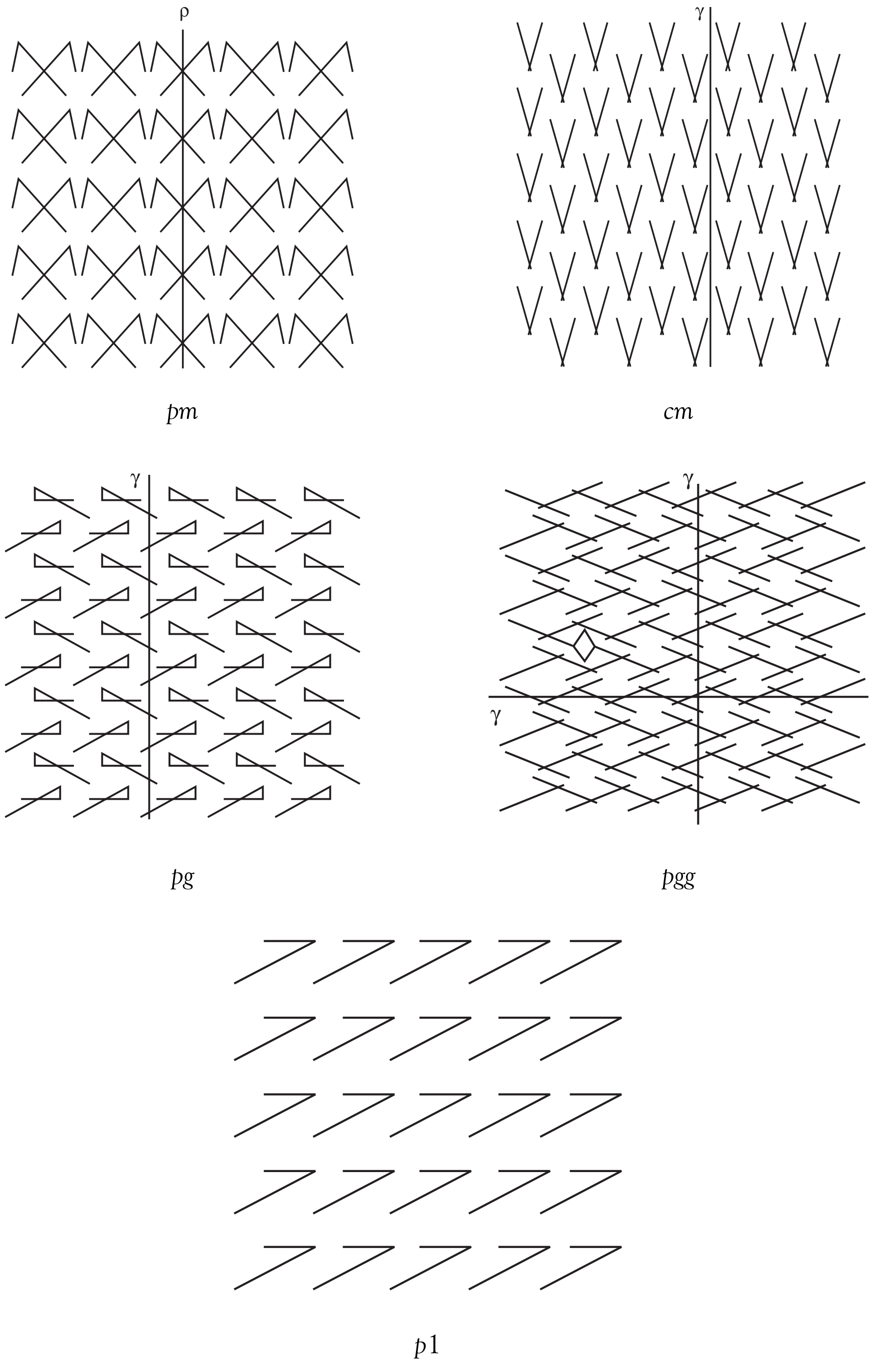


Figure 6.26 *Five wallpaper designs*

EXERCISES 6.5

Determine the crystallographic symbol of each of the following wallpaper designs 1–34. In each case,

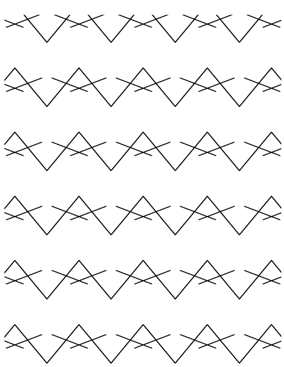
a) Display a rotation of the highest order.

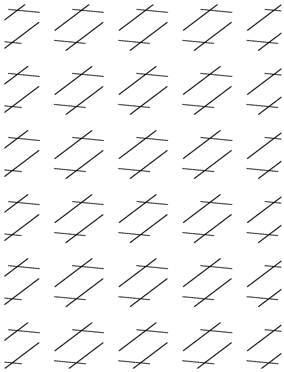
b) Denote the presence of a glide reflection by drawing its axis with an accompanying γ.

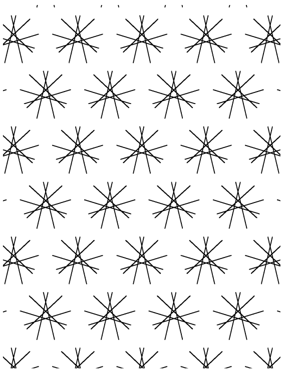
c) Denote the presence of a reflection by drawing its axis with an accompanying ρ.

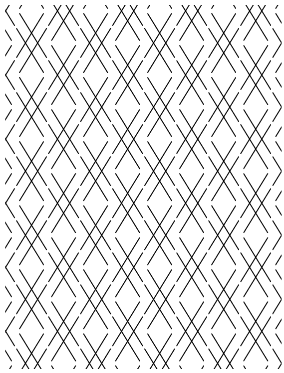
d) Avoid redundancy by only drawing only one axis in any direction.

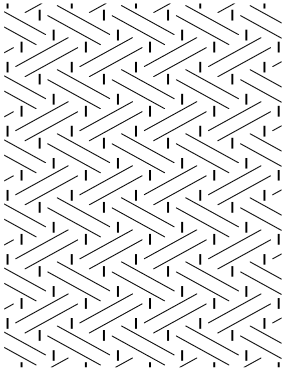
e) In case you have to choose between a ρ and a γ, display the γ.

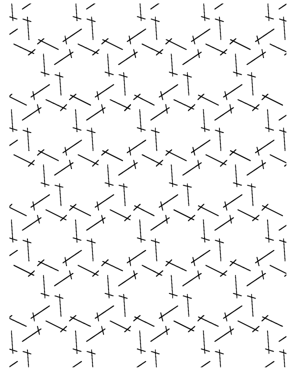

1.

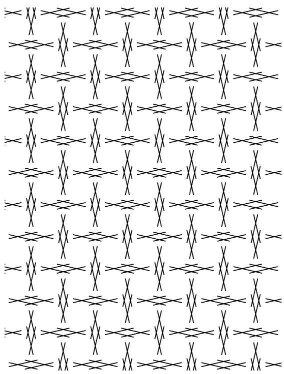

2.

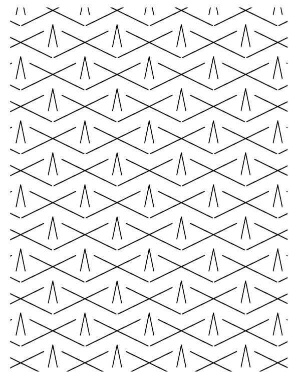

3.

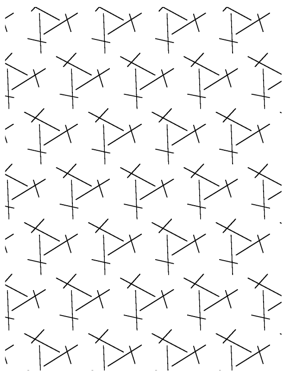

4.

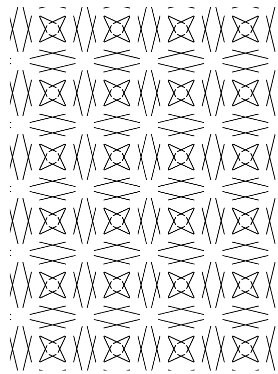
5.


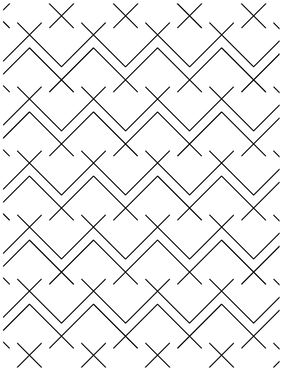
6.


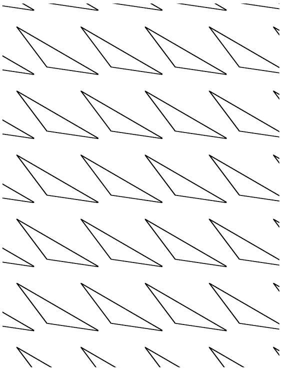
7.


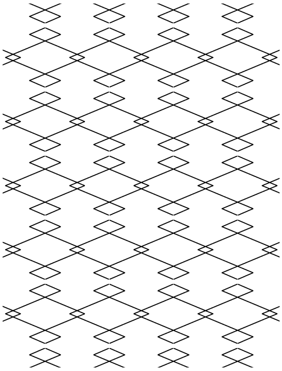
8.


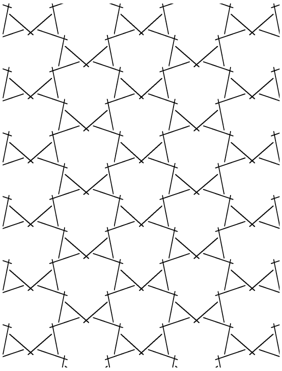
9.


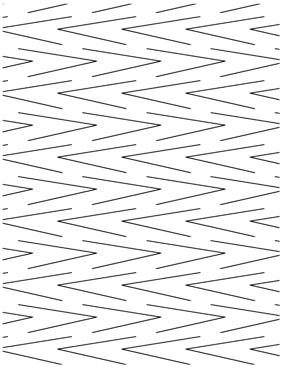
10.


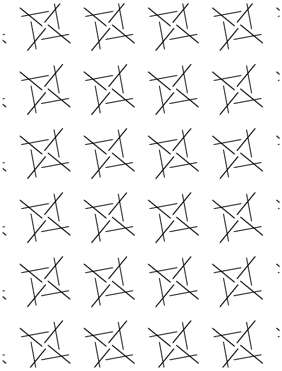
11.


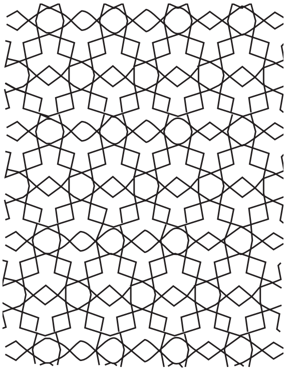
12.


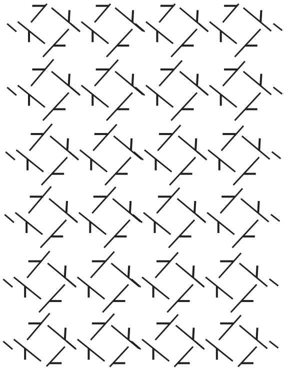
13.


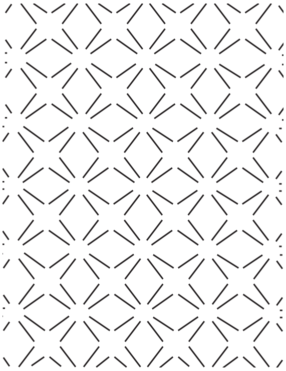
14.


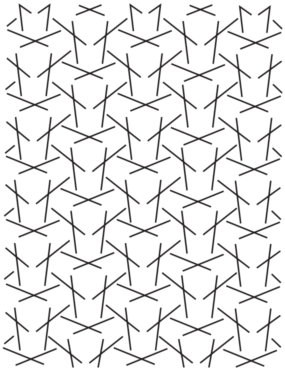
15.


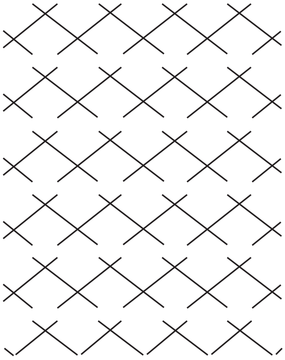
16.


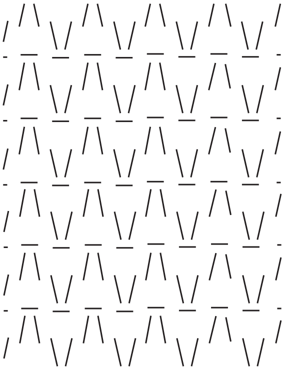

17.

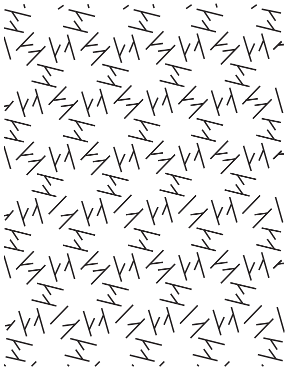

18.


19.

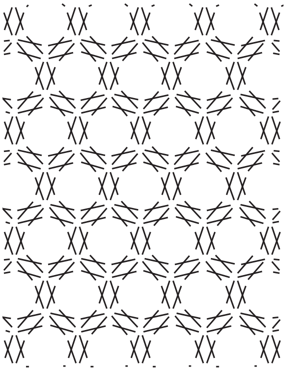

20.


21.

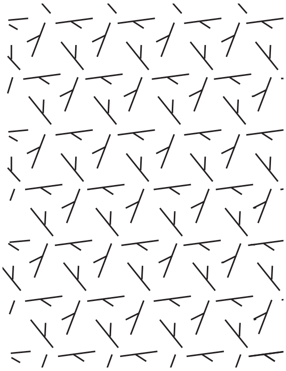

22.



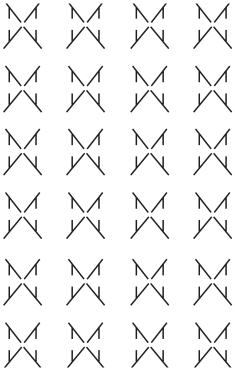
23.



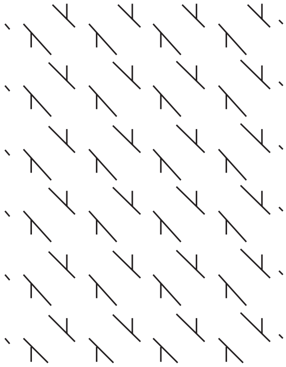
24.



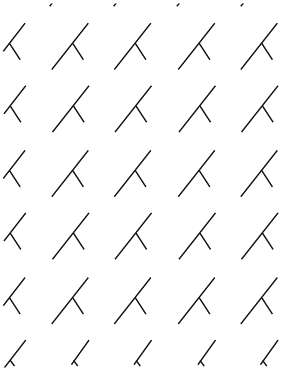
25.



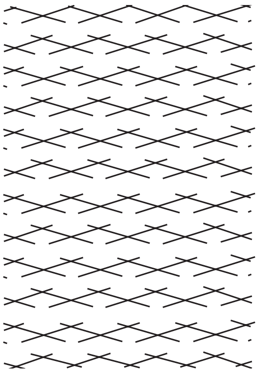
26.

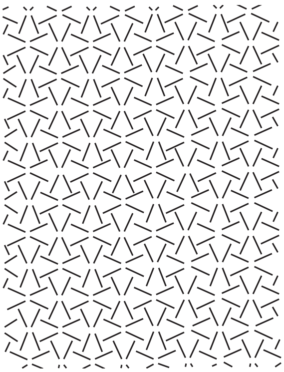


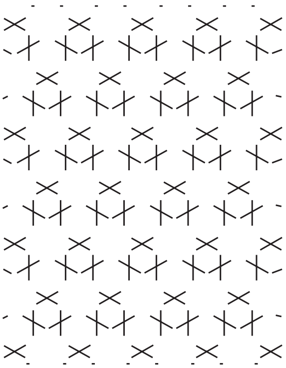
27.

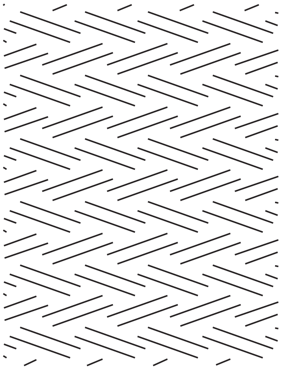


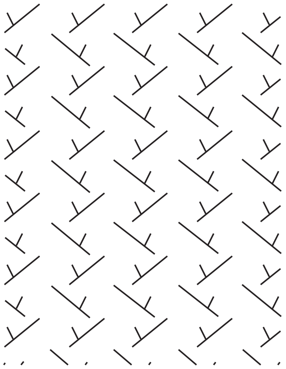
28.

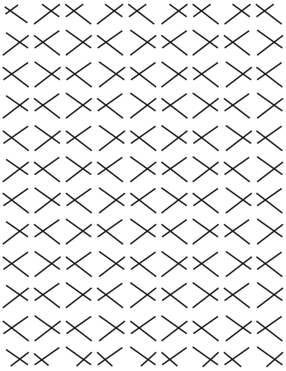
29. 

30. 

31. 

32. 

33. 

34. 

6.6 References and Further Readings

Ausdley, W. and G., *Designs and Patterns from Historic Ornaments*, Dover Publications, New York, 1989.

Budden, F. J., *Fascination of Groups*, Cambridge University Press, Cambridge, 1972.

Farmer, D. W., *Groups and Symmetry: A Guide to Discovering Mathematics*, American Mathematical Society, Washington DC, 1996.

Grünbaum, B., and Shepard, G. C., *Tilings and Patterns*, W. H. Freeman and Co., 1987.

Jones, O., *The Grammar of Chinese Ornament*, Studio Editions, London, 1987.

Martin, G. E., *Transformation Geometry: An Introduction to Symmetry*, Springer-Veralg, New York, 1982.

Schattschneider, D., The Plane Symmetry Groups, Their Recognition and Notation, *Amer. Math. Monthly*, **85**(1978) no. 6, 439–450.

Weyl, H., *Symmetry*, Princeton University Press, Princeton, 1952.