CHAPTER ONE

Systems of Linear Equations

1.1 SOLVING SYSTEMS OF LINEAR EQUATIONS

Do not worry about your difficulties in mathematics. I can assure you that mine are still greater.

—ALBERT EINSTEIN (1879–1955)

Begin at the beginning and go until you come to the end: then stop.

—LEWIS CARROLL (1832–1889),
ALICE'S ADVENTURES IN WONDERLAND

The term *linear algebra* applies to a branch of mathematics that studies vectors, matrices, vector spaces, and systems of linear equations. It is hoped that the reader has acquaintance with some of these terms. For example, a vector with three components looks like $(2.5, 3.7, -5.1)$, while a 2×3 matrix looks like this:

$$\begin{bmatrix} 4.1 & -3.2 & 5.4 \\ 1.3 & 2.0 & -5.1 \end{bmatrix}$$

These two building blocks, vectors and matrices, can produce systems of linear equations, and these in turn can often model an applied problem from the real world.

The subject of linear algebra was already being studied sporadically in ancient times, as is known from surviving manuscripts. But the subject blossomed in the early 1800s and thus is approximately 200 years old. It is much younger than calculus, which was already thriving at the time of Newton and Leibniz, in the 1700s.

In this book you will find many examples illustrating how some computational problem originating in engineering, physics, finance, or economic planning (and in other disciplines) becomes a fully understood type of problem in linear algebra, and therefore yields easily to standard techniques already available. In many cases, such problems can be solved by means of well-tested and documented computer software.

Linear Equations

This topic is basic to much of what comes later. We begin our study by discussing a single linear equation containing two variables.

EXAMPLE 1 Consider the equation $7x - 3y = 21$, which represents a line in the xy-plane. Where does this line cross the two axes?

SOLUTION The question is: What points of the form $(x, 0)$ and $(0, y)$ are on this line? For the first point, we let $y = 0$, and the resulting equation is $7x = 21$. Thus, we have $x = 3$ and the point sought is $(3, 0)$. For the second point, we let $x = 0$ and the equation now reads $-3y = 21$. Hence, we have $y = -7$ and the point wanted is $(0, -7)$. These points are the **intercepts** of the line. See Figure 1.1. ■

The **point–slope form** of a line is

$$y = mx + b$$

where m is the **slope** and b is the **intercept** on the y-axis. From Example 1, the line $7x - 3y = 21$ can be written in the point-slope form as $y = \frac{7}{3}x - 7$.

EXAMPLE 2 Use the line described in Example 1. Are the points $(-3, -14)$, $(3, 1)$, and $(6, 7)$ on the line?

SOLUTION In each case, one can substitute the coordinates in the equation $7x - 3y = 21$ to see whether the equation is satisfied. For the first point, we calculate $(7)(-3) - (3)(-14) = 21$, for the second point $(7)(3) - (3)(1) = 18$, and for the third point $(7)(6) - (3)(7) = 21$. Hence, the first and third points are on the line but the second is not. See Figure 1.1. ■

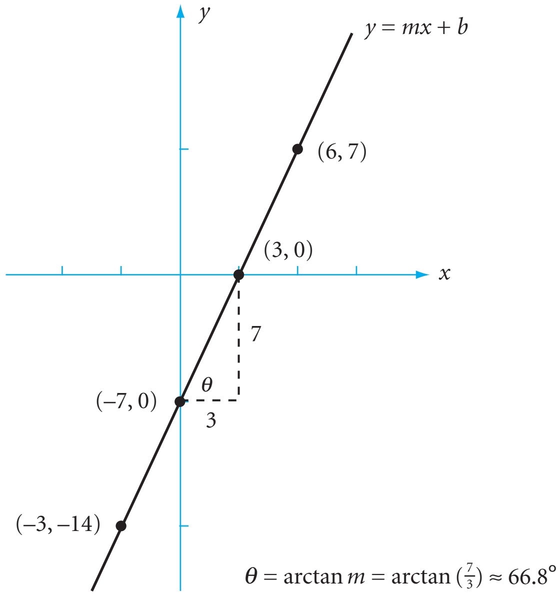

FIGURE 1.1 A point–slope form of a line in $\mathbb{R}^2$.

For two points (x_0, y_0) and (x_1, y_1), the **two-point form** of the line through these points is

$$y - y_0 = m(x - x_0) \quad \text{where} \quad m = \frac{y_1 - y_0}{x_1 - x_0}$$

For example, using the two points in Example 2, we obtain the two-point form as $y - 7 = \frac{7}{3}(x - 6)$.

The equation discussed above is called a **linear equation** precisely because its graph is a line, and the word **linear** derives from the word *line.*

We extend the meaning of a **linear equation** to encompass one of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \qquad \text{or} \qquad \sum_{j=1}^{n} a_jx_j = b$$

involving n variables. Here we have named the variables x_1, x_2, and so on because we need the flexibility of handling any number of variables, even hundreds of thousands! The variables x_j may (in some contexts) also be called **unknowns**. The second form of writing the equation employs standard summation notation: the variable j runs through the integers 1 to n, and we are to take the sum of all the resulting terms, a_jx_j. When we use summation notation, the variable index (j in the preceding equation) can be almost any convenient variable, such as i, j, k, μ, or ν. However, one must be careful to avoid any conflict with letters already used in a different context. For example, $\sum_{i=1}^{n} x_i = y_i$ is obviously wrong!

Systems of Linear Equations

We are rarely interested in only one such equation in isolation. We usually encounter *systems* of linear equations. We must stretch our notation a little bit further and settle upon the following standard formulation. A completely general system of m linear equations with n unknowns (or **variables**) has equations of this form

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{i,n-1}x_{n-1} + a_{in}x_n = b_i \qquad (1 \leq i \leq m)$$

which is the exact form of the ith generic equation in the system. Sometimes a comma is needed to separate the two subscripts on the letter a if it is not clear otherwise. Each equation in the system involves the same set of variables, $x_1, x_2, x_3, \ldots, x_n$. The entire system can be written succinctly with summation notation:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (1 \leq i \leq m)$$

The symbolism on the left of this equation means a sum of terms, each of the form $a_{ij}x_j$. The index j runs over the integer values from 1 to n. Off to the side we see in parentheses an indication that the index i also runs through a set of integers, in this case $i = 1, 2, 3$, up to and including m. Here and elsewhere, we often expect i and j to be restricted to nonnegative integer values.

An important concept that we will return to in Section 1.2 is the consistency of systems of linear equations.

DEFINITION

A system of equations is **consistent** *if it has at least one solution, and* **inconsistent** *if it has no solution.*

Next, we consider a system of two linear equations in two unknowns:

$$L_1 : -x + y = 1 \qquad L_2 : 2x + y = 4$$

These two equations correspond to two lines L_1 and L_2 in $\mathbb{R}^2$. Adding 2 times the first equation to the second equation produces $3y = 6$ or $y = 2$.

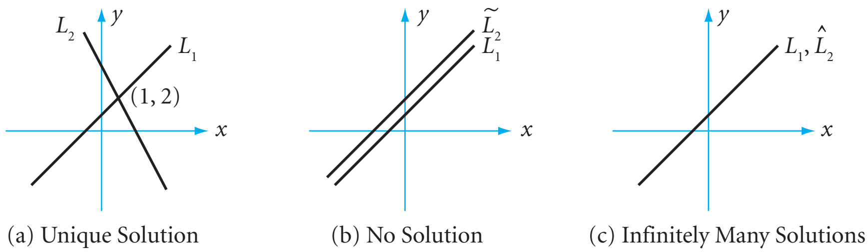

FIGURE 1.2 Different cases of two lines in $\mathbb{R}^2$.

Substituting this value into the first equation reveals that $x = 1$. Plots of these two lines are shown in Figure 1.2(a), and we see that they intersect at the point $(1, 2)$. In this case, there is exactly *one* solution.

Modifying the second equation, we consider the system

$$L_1 : -x + y = 1 \qquad \widetilde{L}_2 : 2x - 2y = 4$$

Dividing the second equation by -2, we find that this pair of equations is *inconsistent*, requiring that $-x + y = 1$ and $-x + y = -2$. This means that there is *no* solution. As shown in Figure 1.2(b), these two lines are parallel and therefore do not intersect at all.

Again, modifying the second equation slightly leads us to the third case:

$$L_1 : -x + y = 1 \qquad \widehat{L}_2 : 2x - 2y = -2$$

Dividing the second equation by -2, we find that these equations are now duplicates of each other. Letting $y = 1$, we find $x = 0$. Letting $y = 3$, we have $x = 2$, and so on. Giving y *any* value, we have $x = -1 + y$. Consequently, there are *infinitely many* solutions in this case. As shown in Figure 1.2(c), there is now *only* one line and we can think of the two equations as having graphs that lie on top of each other.

EXAMPLE 3 A system of four equations in three unknowns is exemplified by

$$\begin{cases} 3x_1 - 2x_2 + 5x_3 = 7 \\ x_1 + 4x_2 - 3x_3 = 7 \\ 6x_1 - 4x_2 + 2x_3 = -2 \\ x_1 + 2x_2 + x_3 = 9 \end{cases}$$

Is the point $(1, 3, 2)$ a solution of this system?

SOLUTION It is simply a matter of putting the numerical values $x_1 = 1$, $x_2 = 3$, and $x_3 = 2$ into each of the four equations and seeing that the equations are indeed satisfied by the given numbers. We have not revealed how $(1, 3, 2)$ was obtained. Indeed, some effort would have to be invested to discover this solution. However, verifying a purported solution is trivial in contrast. It requires only a substitution and a bit of arithmetic. ■

In Example 3, the system of equations is a textbook problem. It is not *typical* of one that would arise in an application, where the data given (i.e., the numbers a_{ij} and b_i) are rarely integers (whole numbers). In this text most of the examples and problems employ integers, for simplicity.

General Systems of Linear Equations

Let us return to the general system of m linear equations in n unknowns. The data for this system of equations are all the numbers a_{ij} and b_i. The number a_{ij} is the coefficient of x_j in the ith equation. In a typical problem, all these coefficients would be given to us numerically. The numbers b_i on the righthand side would also be given.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Then the challenge is to find the values of the *unknowns* $x_1, x_2, \ldots, x_n$ that make the equations true. The coefficient data and the list of unknowns can be exhibited in a number of ways. Consider these four arrays:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

All of these arrays are examples of **matrices**. (The plural form of the word *matrix* is *matrices*.) The middle two are also examples of **column vectors**.

In this context of a system of equations, the four matrices are called, respectively, the **coefficient matrix**, the **vector of unknowns**, the **righthand-side vector**, and the **augmented matrix**. If we call the coefficient matrix **A**, the righthand-side vector **b**, and the unknown vector **x**, the system of equations can be expressed as

$$\mathbf{Ax} = \mathbf{b}$$

This formalism will be correct if we define the product **Ax** appropriately. This comes later.

The matrix **A** displayed in detail earlier is called an $m \times n$ matrix because it has m rows and n columns. The rows are the horizontal arrays inside **A** and the columns are the vertical arrays inside **A**. We always give the number of rows first and the number of columns second in describing the dimensions of a matrix. Hence, we do *not* call **A** an $n \times m$ matrix because that would be a matrix of a different shape, if $n \neq m$. The indices on the letter a, such as a_{ij}, tell us that the number being referenced is in row i and column j. One can call the first index i the *row index*, and the second j the *column index*. Thus, for example, a_{pq} is the element in row p and column q. These traditions must be followed so that we can understand each other when speaking of matrices!

Gaussian Elimination

It is possible to use a process called **Gaussian elimination** to solve any system of linear equations that has a solution.[1]

EXAMPLE 4 Solve this system of linear equations:

$$\begin{cases} 3x_1 + 2x_2 - 5x_3 = -1 \\ \phantom{3x_1 + {}} 4x_2 + x_3 = 14 \\ \phantom{3x_1 + 4x_2 +{}} -2x_3 = -4 \end{cases}$$

[1] This name honors Johann Karl Friedrich Gauss (1777–1855), one of the greatest mathematicians. In elementary school he demonstrated his mathematical potential and amazed his teachers by inventing a simple method for summing an arithmetic series. Namely, one multiplies the number of terms by the average of the smallest and largest terms. In the subject of linear algebra, when he was 18, Gauss invented the method of least squares—a topic taken up in Sections 2.2 and 7.2 of this book. Also, he was the first to prove the **Fundamental Theorem of Algebra**: Every nonconstant polynomial assumes the value 0 at some point in the complex plane.

SOLUTION We observe that this problem has a certain structure that can be exploited in arriving at the solution: namely, the last equation can be solved at once to obtain $x_3 = 2$. Then, with x_3 known, we can solve the second equation for x_2. To do so, write it as $4x_2 + 2 = 14$, and solve for $x_2 = 3$. Finally, with x_2 and x_3 known, we can compute x_1 from the first equation: $3x_1 + (2)(3) + (-5)(2) = -1$, and $x_1 = 1$. We write the solution as $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ or $\mathbf{x} = (1, 3, 2)$; it is a point in three-space (denoted by $\mathbb{R}^3$). ■

It may appear that the problem in Example 4 is artificial, since in practice we do not expect systems of linear equations to have the structure of which we took advantage. However, as we shall see, there is an algorithm for turning a system of equations into the so-called **triangular** form present in Example 4.

EXAMPLE 5 For a concrete example of modest size to illustrate the techniques for solving systems of linear equations, we use this special case:

$$\begin{cases} 3x_1 + 2x_2 = 4 \\ 9x_1 + 7x_2 = 17 \end{cases}$$

SOLUTION The basic operation that can be used over and over again in solving linear systems is the *addition* of a multiple of one equation to another equation. For reasons that will become clear later, we call this a **replacement operation**. In the example, let us add -3 times the first equation to the second. The result is

$$\begin{cases} 3x_1 + 2x_2 = 4 \\ 0x_1 + 1x_2 = 5 \end{cases}$$

In this process, the first equation itself was not changed, although it played a role in this first step. At this stage in the solution process, we have a choice. First, we can see that the new second equation can be solved immediately to get $x_2 = 5$. Then, as in Example 4, we can solve the first equation for x_1, since x_2 is 5. This yields $x_1 = -2$. The alternative way of proceeding is to carry out another replacement operation to produce a zero coefficient of x_2 in the first equation. In fact, we should add to the first equation -2 times the second equation, getting

$$\begin{cases} 3x_1 + 0x_2 = -6 \\ 0x_1 + 1x_2 = 5 \end{cases}$$

The solution is now clear: $x_1 = -2$ and $x_2 = 5$. (Of course, we could divide the first equation by 3 so that the solution values appear on the righthand side. This is a **scale operation**.) There is an independent manner of verifying the work: Simply substitute the purported solution into the original system of equations to see whether it is actually a solution. The work of doing so yields $(3)(-2) + (2)(5) = 4$ and $(9)(-2) + (7)(5) = 17$. It is a good habit to find independent ways of verifying a solution—that is, methods different from simply checking the steps that led to a solution in the first place. ■

The process we have just illustrated is often called **Gaussian elimination**. The process is also called the **row-reduction algorithm**. The process whereby the system of equations produces explicit values of each variable is called **Gauss–Jordan elimination** in honor of Gauss and Wilhelm Jordan.[2]

Elementary Replacement and Scale Operations

In the examples, we have been using two elementary row operations: a **replacement** operation of adding a multiple of one equation to another equation, and a **scale** operation of multiplying an equation by a nonzero scalar. How can we be sure that, when we transform a system in the way that we did, we do not introduce spurious solutions or lose genuine solutions? It is simply that one can *add equal quantities to equal quantities* to obtain further equalities, and the process can be reversed. For example, if we write the original pair of equations as

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[2] Wilhelm Jordan (1842–1899) is remembered for making improvements in the stability of the Gaussian elimination algorithm when it is applied to least squares problems. In the *Gauss-Jordan elimination* procedure for solving systems of linear equations, *Jordan* is the geodesist Wilhelm Jordan. [Some people have made the mistake of crediting Camille Jordan (1838–1922) in this context. In the *Jordan normal form of a matrix*, it is indeed Camille Jordan who is to be credited.] In the simple Gauss procedure (Gaussian elimination), row operations are used to produce an upper triangular coefficient matrix, whereas in the Gauss–Jordan computation, the row operations are designed to lead to the identity matrix on the left, and the solution vector on the right. Wilhelm Jordan had a brilliant career as a master surveyor and was involved in surveying large areas of Germany. His textbook *Handbook of Geodesy*, in German, went through five editions and was translated into French, Italian, and Russian. See Althoen and McLaughlin [1987].

$$\begin{cases} E_1(\mathbf{x}) = 0 \\ E_2(\mathbf{x}) = 0 \end{cases}$$

then we can proceed to

$$\begin{cases} E_1(\mathbf{x}) = 0 \\ \alpha E_1(\mathbf{x}) + E_2(\mathbf{x}) = 0 \end{cases}$$

where α is any real number—that is, a scalar. Thus, when the first pair of equations is true, the second pair must also be true. In other words, an $\mathbf{x}$ that satisfies the first pair will also satisfy the second pair. Furthermore, we can get back to the first pair by applying a similar process to the second pair: to the second equation in the second pair, we add $-\alpha E_1(\mathbf{x})$. One can say that the row operation we are talking about here is **reversible** by another row operation of the same type. This shows that any solution of the second pair of equations must satisfy the first pair. These two parallel assertions establish that solutions are neither created nor destroyed in the process we are using. To emphasize:

1. Every solution of the first pair of equations satisfies the second pair.
2. Every solution of the second pair of equations satisfies the first pair.

If two systems of m linear equations in n unknowns have precisely the same set of solutions, we can get the solutions to the first system by solving the second, or vice versa. This simple idea is at the heart of our procedure for solving systems of linear equations. The remarks in the preceding paragraph establish that if one system of equations is obtained from another by a sequence of the permitted row operations, then the resulting two systems of equations have precisely the same set of solutions. In other words, the steps that we use do not change the set of solutions. With this in mind, we aim for a simple set of equations derived from the one with which we started. The solutions of the simple system are exactly the solutions of the original. This is the grand strategy, which has a number of variations. As matters stand, two people could work on a system of equations and produce two different sequences of simplified systems. The solutions of the simplified systems, in each of the two sequences, should be the same, however!

Row-Equivalent Pairs of Matrices

It should be pointed out that the work needed to solve the previous system of equations requires only that we keep track of the numerical data. There is no need to write the names of the variables in each step, or the equals sign. Thus, we can set up the data in successive arrays like this:

$$\left[\begin{array}{cc|c} 3 & 2 & 4 \\ 9 & 7 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 2 & 4 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 0 & -6 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

In this display, the symbol $\sim$ means that the matrices on either side of the symbol are connected by allowable row operations. In English, we say that the two matrices are **row equivalent** to each other.

Some readers may wish to describe each of the steps with the following notation:

$$\begin{array}{c} \rightarrow \\ -3 \end{array} \left[\begin{array}{cc|c} 3 & 2 & 4 \\ 9 & 7 & 17 \end{array}\right] \sim \begin{array}{c} -2 \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 3 & 2 & 4 \\ 0 & 1 & 5 \end{array}\right] \sim \tfrac{1}{2} \left[\begin{array}{cc|c} 3 & 0 & -6 \\ 0 & 1 & 5 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

Here we have written to the left of these matrices an arrow $\rightarrow$ indicating the so-called **pivot row** and a number indicating the **multiplier** next to the *target* row. A multiple of the pivot row is added to the target row. This is helpful in recalling what was done. We rarely include these symbols in the text, but the reader may wish to add them.

The rectangular arrays of data are again *matrices*. A **matrix** can be any rectangular array of real numbers. In the preceding example, we are using 2×3 matrices, meaning that there are two horizontal **rows** and three vertical **columns**. The symbol $\sim$ indicates that we have proceeded from one matrix to another by one or more **row operations** of the permitted type: addition of a multiple of one row to another row. Later we shall add further row operations to our arsenal, and the notation $\mathbf{A} \sim \mathbf{B}$ will mean that each of the matrices $\mathbf{A}$ and $\mathbf{B}$ can be obtained from the other by applying one or more allowable row operations. We say in this case that the two matrices are **row equivalent** to each other.

This type of relation occurs often in mathematics, especially in linear algebra. A formal definition follows. In this definition we have deliberately chosen a symbol, $\bigstar$, that is not likely to conflict with other notation.

DEFINITION

An **equivalence relation** *on a set of entities is a relation that we denote here by the symbol* $\bigstar$*. It must satisfy these three conditions:*

- $p \bigstar p$. (**reflexive**)
- *If* $p \bigstar q$*, then* $q \bigstar p$. (**symmetric**)
- *If* $p \bigstar q$ *and* $q \bigstar r$*, then* $p \bigstar r$. (**transitive**)

An example of an equivalence relation is $\mathbf{A} \sim \mathbf{B}$. To see why, think of the row operations used. Clearly, $\mathbf{A} \sim \mathbf{A}$ and the reflexive property holds. Why is the symmetric property true for a row-equivalence relation? The row operations that take us from $\mathbf{A}$ to $\mathbf{B}$ are reversible, and the reverse operations lead from $\mathbf{B}$ back to $\mathbf{A}$. For the transitive property, if $\mathbf{A} \sim \mathbf{B}$ and $\mathbf{B} \sim \mathbf{C}$, then $\mathbf{A} \sim \mathbf{C}$ because the respective row operations from $\mathbf{A}$ and $\mathbf{B}$ can be done one after another on $\mathbf{A}$.

When solving textbook problems (which often have integer data), one can avoid divisions and fractions in some row reductions. For example, consider this row reduction where we begin with row two as the first pivot row:

$$\begin{matrix} -1 \\ \rightarrow \\ \end{matrix} \left[\begin{array}{rr|r} 3 & -1 & 2 \\ 2 & 5 & -4 \\ 7 & -25 & 1 \end{array}\right] \sim \begin{matrix} \rightarrow \\ -2 \\ -7 \end{matrix} \left[\begin{array}{rr|r} 1 & -6 & 6 \\ 2 & 5 & -4 \\ 7 & 25 & 1 \end{array}\right] \sim \begin{matrix} \\ \rightarrow \\ -1 \end{matrix} \left[\begin{array}{rr|r} 1 & -6 & 6 \\ 0 & 17 & -16 \\ 0 & 17 & -41 \end{array}\right]$$

$$\sim \left[\begin{array}{rr|r} 1 & -6 & 6 \\ 0 & 17 & -16 \\ 0 & 0 & 15 \end{array}\right]$$

Here there is *no* solution. Indeed, the third equation states that $0x+0y = 15$. Recall that a system of linear equations is inconsistent when it has no solution, and consistent when it has one or more solutions. These concepts are studied further in Section 1.2. In the preceding example, each pair of equations has a unique solution, but there exists no point satisfying all three equations simultaneously.

We have inserted a vertical line in a matrix to separate the coefficient matrix from the righthand side, if the matrix is the augmented matrix of a system of equations. On the right side of this line, we have the righthand side of the original system of equations. We will encounter situations where there are multiple columns to the right of this line, and therefore it is a good practice to place this vertical line in any augmented matrix.

If we have several systems with the same lefthand side but different righthand sides, we can use an augmented matrix to solve them simultaneously. For example, we can solve these two systems

$$\begin{cases} 3x_1 + 2x_2 = 1 \\ 9x_1 + 7x_2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} 3x_1 + 2x_2 = 0 \\ 9x_1 + 7x_2 = 1 \end{cases}$$

by row-reducing the following augmented matrix, in which there are two righthand sides:

$$\begin{matrix} \rightarrow \\ -3 \end{matrix} \left[\begin{array}{rr|rr} 3 & 2 & 1 & 0 \\ 9 & 7 & 0 & 1 \end{array}\right] \sim \begin{matrix} -2 \\ \rightarrow \end{matrix} \left[\begin{array}{rr|rr} 3 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{rr|rr} 3 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

The solutions are $x_1 = \frac{7}{3}$, $x_2 = -3$ for the first system and $x_1 = -\frac{2}{3}$, $x_2 = 1$ for the second system. This procedure can be useful later in computing inverses of matrices. (See Section 3.2.)

EXAMPLE 6 Another example, requiring a little more thought, calls for solving the following system:

$$\begin{cases} 7x_1 + 5x_2 = 83/6 \\ 37x_1 + 7x_2 = 361/6 \end{cases}$$

SOLUTION Our method is the same, and we add a multiple of the first equation to the second equation to create a 0 where the number 37 stands. Denote the unknown multiplier by α. Then we want $37 + \alpha 7 = 0$, and the multiplier should be $\alpha = -37/7$. The resulting system is

$$\begin{cases} 7x_1 + 5x_2 = 83/6 \\ 0x_1 - (136/7)\, x_2 = -272/21 \end{cases}$$

In the same way as before, we now decide that 35/136 times the second equation should be added to the first. The result of doing so is

$$\begin{cases} 7x_1 + 0x_2 = 21/2 \\ 0x_1 - (136/7)\, x_2 = -305/42 \end{cases}$$

The solution is therefore $x_1 = \frac{3}{2}$ and $x_2 = \frac{2}{3}$. We summarize the work in matrices like this:

$$\left[\begin{array}{rr|r} 7 & 5 & 83/6 \\ 37 & 7 & 361/6 \end{array}\right] \sim \left[\begin{array}{lr|r} 7 & 5 & 83/6 \\ 0 & -136/7 & -272/21 \end{array}\right]$$

$$\sim \left[\begin{array}{lr|r} 7 & 0 & 21/2 \\ 0 & -136/7 & -272/21 \end{array}\right]$$

■

EXAMPLE 7 Given the following equivalent matrices:

$$\left[\begin{array}{rrr|r} 5 & 3 & 0 & 21 \\ 1 & 3 & -1 & 15 \\ -2 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 0 & -12 & 5 & -54 \\ 1 & 3 & -1 & 15 \\ 0 & 6 & -3 & 31 \end{array}\right] \sim \left[\begin{array}{rrr|r} 0 & 0 & -1 & 8 \\ 1 & 3 & -1 & 15 \\ 0 & 6 & -3 & 31 \end{array}\right]$$

What is the solution?

SOLUTION In this example, further reduction can be carried out, but observe that the system of equations can be easily solved by using the work already done. To determine x_3, use the first equation (in the final array), which asserts that $x_3 = -8$. With x_3 in hand, we now use the third equation, which indicates that $6x_2 - 3x_3 = 31$. This becomes $6x_2 + 24 = 31$; therefore, $x_2 = \frac{7}{6}$. Lastly, use equation two, which asserts that $x_1 + 3x_2 - x_3 = 15$. From this, we have $x_1 + 3(\frac{7}{6}) - (-8) = 15$ and $x_1 = \frac{7}{2}$. The procedure we have just illustrated is the one usually used in mathematical software. In brief, we reduce the matrix so that there is one equation containing only one unknown and another equation containing that unknown and one other, and so on. The system is readily solved by starting with the equation having only one unknown, and proceeding through the whole system one equation at a time.

In the preceding example, further row reduction leads to

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 8 \\ 2 & 6 & -2 & 30 \\ 0 & 6 & -3 & 31 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & -1 & 8 \\ 2 & 0 & 1 & -1 \\ 0 & 6 & -3 & 31 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & -1 & 8 \\ 2 & 0 & 0 & 7 \\ 0 & 6 & 0 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & -8 \\ 1 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 7/6 \end{array}\right]$$

Again, we obtain $x_1 = \frac{7}{2}$, $x_2 = \frac{7}{6}$, and $x_3 = -8$. ■

The row operations that we have been exploiting are (1) addition to one row of a multiple of another row and (2) multiplying a row by a nonzero constant. These two operations are sufficiently powerful to solve any system of linear equations by repeated application. (We are not stopping to prove this fact.) Example 7 illustrates how this is done. Here is another example on which to test yourself.

EXAMPLE 8 Solve this system of linear equations by repeated use of the row operation illustrated in the two preceding examples:

$$\begin{cases} 35x_1 + 6x_2 - 7x_3 = 15 \\ -90x_1 - 15x_2 + 21x_3 = -40 \\ 25x_1 + 3x_2 - 7x_3 = 11 \end{cases}$$

SOLUTION We systematically create zeros in strategic positions. Here is one sequence of reductive steps. Start by subtracting row three from row one. Then add 3 times row three to row two. Next add 2 times row one onto row two. Then add −3 times row two to row three. Finally, add −2 times row two to row one and −3 times row two to row three.

$$\left[\begin{array}{rrr|r} 35 & 6 & -7 & 15 \\ -90 & -15 & 21 & -40 \\ 25 & 3 & -7 & 11 \end{array}\right] \sim \left[\begin{array}{rrr|r} 10 & 3 & 0 & 4 \\ -90 & -15 & 21 & -40 \\ 25 & 3 & -7 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 10 & 3 & 0 & 4 \\ -15 & -6 & 0 & -7 \\ 25 & 3 & -7 & 11 \end{array}\right] \sim \left[\begin{array}{rrr|r} 10 & 3 & 0 & 4 \\ 5 & 0 & 0 & 1 \\ 15 & 0 & -7 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 0 & 3 & 0 & 2 \\ 5 & 0 & 0 & 1 \\ 0 & 0 & -7 & 4 \end{array}\right]$$

Therefore, the solution is $x_1 = \frac{1}{5}$, $x_2 = \frac{2}{3}$, and $x_3 = -\frac{4}{7}$. ■

In the preceding example, we have illustrated the judicious use of row operations to avoid dealing with fractions. Conversely, some may prefer to follow the systematic approach of using the first row as the pivot row and creating zeros in the first column below the first row. Next, using the second row as the pivot row, one can create zeros in the second column below the second row, and so on.

Elementary Row Operations

Next on our agenda is the introduction of another of the **elementary row operations**, (3) the multiplication of a row by a nonzero scalar. This operation is a **scaling** of an equation. With this operation, we can further reduce some of the preceding matrices as follows:

$$\left[\begin{array}{rr|r} 7 & 0 & 21/2 \\ 0 & -136/7 & -272/21 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 3/2 \\ 0 & 1 & 2/3 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 0 & 0 & -1 & 8 \\ 1 & 0 & 0 & 7/2 \\ 0 & 6 & 0 & 7 \end{array}\right] \sim \left[\begin{array}{rrr|r} 0 & 0 & 1 & -8 \\ 1 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 7/6 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 0 & 3 & 0 & 2 \\ 5 & 0 & 0 & 1 \\ 0 & 0 & -7 & 4 \end{array}\right] \sim \left[\begin{array}{rrr|r} 0 & 1 & 0 & 2/3 \\ 1 & 0 & 0 & 1/5 \\ 0 & 0 & 1 & -4/7 \end{array}\right]$$

As with the first type of row operation (adding a multiple of one row onto another row), we should be sure that we do not alter the solutions when we use this new row operation. But this fact is obvious: If we look at only one equation, $E(\mathbf{x}) = 0$, then this is equivalent to $\alpha E(\mathbf{x}) = 0$, provided that $\alpha \neq 0$. This last condition is necessary to get back to $E(\mathbf{x}) = 0$ from $\alpha E(\mathbf{x}) = 0$; of course, we must multiply by $1/\alpha$.

Without introducing any really new row operations, we can now perform **interchanges** or **swaps** among the rows in a system of equations. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two rows. With a succession of row operations of the two types already described, we can execute a *swap*, that is an *interchange* of two rows:

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \sim \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_1 + \mathbf{r}_2 \end{bmatrix} \sim \begin{bmatrix} -\mathbf{r}_2 \\ \mathbf{r}_1 + \mathbf{r}_2 \end{bmatrix} \sim \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 + \mathbf{r}_2 \end{bmatrix} \sim \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \end{bmatrix}$$

Despite the fact that only two types of row operations are needed, it is conventional to define three types of **elementary row operations**:

1. **(replacement)** Add a multiple of one row to another.
2. **(scale)** Multiply a row by a nonzero factor.
3. **(swap)** Interchange a pair of rows.

The rows change positions only in a swap operation. Remember that in any proof involving row operations, the swapping process can be performed with four replacement and scaling operations. Thus, only replacement and scaling operations are essential in proofs.

If it is desired to describe the row operations being used on a matrix, the following notation is suggested:

replacement	$\mathbf{r}_i \leftarrow \mathbf{r}_i + a\,\mathbf{r}_j$	$(i \neq j,\ a$ is a scalar$)$
scale	$\mathbf{r}_i \leftarrow c\,\mathbf{r}_i$	(scalar $c \neq 0$)
swap	$\mathbf{r}_i \leftrightarrow \mathbf{r}_j$	

In this description, the rows are $\mathbf{r}_1, \mathbf{r}_2$, and so forth. The replacement operation is the adding of a multiple of one row onto another row. The scaling operation is simply multiplying a row by a nonzero constant. The swap operation is the interchanging of two rows. (A similar notation is common in computer science and computer programming.) To illustrate, we use Example 5 (p. 8). The operations needed to reduce the given matrix are $(\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1)$, $(\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 2\mathbf{r}_2)$, and $(\mathbf{r}_1 \leftarrow \frac{1}{3}\mathbf{r}_1)$. Note that the order of performing these steps must be observed.

In computer programs for solving large systems, swapping of rows can be avoided to reduce so-called *data motion*. An index array may be used to keep track of the order of pivot rows. In addition, it is usually not necessary to make the pivot elements equal to unity. High-quality software usually proceeds as we did in Example 7 (p. 13), resulting in one equation having only the term x_3, another equation involving x_2 and x_3, and a third equation involving all three variables.

Reduced Row Echelon Form

For a deeper understanding of linear equations, we often want the simplest form of a system of equations arrived at by use of all three types of row operations. At this stage, we can summarize our work: With suitable row operations, any matrix can be transformed into a special standard form called ***reduced row echelon form***. This form is characterized as follows.

DEFINITION

A matrix is in **reduced row echelon form** *if*

- *All zero rows have been moved to the bottom of the matrix.*
- *Each nonzero row has 1 as its leading nonzero entry, using left-to-right ordering. Each such leading 1 is called a* **pivot**.
- *In each column containing a pivot, there are no other nonzero elements.*
- *The pivot in any row is farther to the right than the pivots in rows above.*

An important theorem, which we prove in Section 1.3 (Theorem 6, p. 79), asserts the following:

THEOREM 1

Every matrix has one and only one reduced row echelon form.

With the help of this theorem, we see that the pivots are uniquely determined by the given matrix.

DEFINITION

A **pivot position** *in a matrix is a location where a leading 1 (a* **pivot**) *appears in the reduced row echelon form of that matrix.*

In general, we do not know the pivot positions until we have found the reduced row echelon form of the matrix, or *any* row echelon form (to be defined on p. 20).

Here are four examples of matrices in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 0 & -7 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

All the 1's in these four matrices happen to be pivots. (See the previous definition to verify this assertion.) Here are four examples of matrices *not* in reduced row echelon form:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An example of the general structure of a matrix in reduced row echelon form is as follows:

$$\begin{bmatrix} 0 & \boxtimes & \times & 0 & 0 & \times \\ 0 & 0 & 0 & \boxtimes & 0 & \times \\ 0 & 0 & 0 & 0 & \boxtimes & \times \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the boxed entries are the pivot positions. The symbol $\times$ designates either a zero or nonzero entry. Notice the staircase pattern of the pivots and zeros, and the fact that above and below each pivot the entries are 0. For an $n \times n$ matrix, the reduced row echelon form may be an **identity matrix**. This matrix is square and is denoted by $\mathbf{I}$ or $\mathbf{I}_n$. Its entries are δ_{ij}, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ when $i \neq j$. This notation, δ_{ij}, is called the **Kronecker delta**.[3] For example, the 5×5 identity matrix $\mathbf{I}_5$ is shown here:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[3] Leopold Kronecker (1823–1891) concentrated his research on the theory of algebraic numbers, and contributed to several new branches of mathematics, viz. group theory and field theory.

EXAMPLE 9 Show how to avoid fractions until the last step in finding the reduced row echelon form for this matrix $\begin{bmatrix} 3 & 2 & -3 & -3 \\ 5 & 1 & 0 & 7 \\ 7 & -4 & 32 & 6 \end{bmatrix}$.

SOLUTION The tactic is to scale two rows so that their leading nonzero entries are the same and then simply subtract one of these rows from the other, creating a zero leading entry in one row.

$$\begin{bmatrix} 3 & 2 & -3 & -3 \\ 5 & 1 & 0 & 7 \\ 7 & -4 & 32 & 6 \end{bmatrix} \sim \begin{bmatrix} 15 & 10 & -15 & -15 \\ 15 & 3 & 0 & 21 \\ 7 & -4 & 32 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 15 & 10 & -15 & -15 \\ 0 & -7 & 15 & 36 \\ 7 & -4 & 32 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & -3 & -3 \\ 0 & -7 & 15 & 36 \\ 7 & -4 & 32 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 21 & 14 & -21 & -21 \\ 0 & -7 & 15 & 36 \\ 21 & -12 & 96 & 18 \end{bmatrix} \sim \begin{bmatrix} 21 & 14 & -21 & -21 \\ 0 & -7 & 15 & 36 \\ 0 & -26 & 117 & 39 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & -3 & -3 \\ 0 & -7 & 15 & 36 \\ 0 & -2 & 9 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 11 & 0 & 0 & 34 \\ 0 & -11 & 0 & 93 \\ 0 & 0 & 11 & -17 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 34/11 \\ 0 & 1 & 0 & -93/11 \\ 0 & 0 & 1 & -17/11 \end{bmatrix}$$

We leave the intermediate steps as an exercise. ■

These results can be confirmed by using the Matlab command `format rat` and `rref(A)`. One can also use `rrefmovie(A)` to see the algorithm working step-by-step. (See the examples on p. 61.)

Fortunately for us, advanced mathematical software systems have built-in commands for computing the reduced row echelon form of matrices. For example, Matlab uses `rref(A)`, Maple has `ReducedRowEchelonForm(A)`, and Mathematica has `RowReduce[A]`. Further examples using these commands are found in the subsections on Mathematical Software at the ends of sections.

Row Echelon Form

A partially reduced form of a matrix is often used; it has three weaker properties.

DEFINITION

A matrix is in **row echelon form** *if*

- *All zero rows have been moved to the bottom.*
- *The leading nonzero element in any row is farther to the right than the leading nonzero element in the row just above it.*
- *In each column containing a leading nonzero element, the entries below that leading nonzero element are 0.*

For example, the matrix below is in row echelon form:

$$\begin{bmatrix} 0 & 3 & 5 & 9 & 6 & 4 & 1 \\ 0 & 0 & 0 & 7 & 6 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

An example of the general structure of a matrix in row echelon form is as follows:

$$\begin{bmatrix} 0 & \boxtimes & \times & \times & \times & \times \\ 0 & 0 & 0 & \boxtimes & \times & \times \\ 0 & 0 & 0 & 0 & \boxtimes & \times \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the boxed entries $\boxtimes$ are the leading nonzero entries in the rows. It is obvious that a row echelon form is obtained with less work than is required for the reduced row echelon form. Notice the staircase pattern of the pivot positions. For some questions about a matrix (such as its rank), the row echelon form gives the answers more quickly. The reduced row echelon form of a matrix is unique, whereas a matrix may have many row echelon forms. Thus, one may ask for *the* reduced row echelon form of a matrix or *a* row echelon form. A proof of the uniqueness of the reduced row echelon form is presented in Section 1.3 (Theorem 6, p. 79). The concepts of reduced row echelon form and row echelon form are independent of whether the matrix is an augmented matrix. We may ignore the vertical line in an augmented matrix when deciding whether it is in row echelon or reduced row echelon form.

Intuitive Interpretation

There is an intuitive interpretation of a system of linear equations that helps one cope with some of the mysteries in linear algebra.

Let us start with a vector $\mathbf{x} = (x_1, x_2)$ that is free to roam all over $\mathbb{R}^2$. The two variables x_1 and x_2 are not connected. It is often said in such a situation that the point has *two degrees of freedom.*

What happens when we impose a single linear condition on this point? Suppose, for example, that we require $x_1 + 3x_2 = 7$. The point now has only *one degree of freedom.* If we start with a point that satisfies this equation and then decide to allow the point to move, while still obeying that equation, it can move only in such a way that the equation $x_1 + 3x_2 = 7$ remains true. As we know, this means that the points must lie on a line. We can start with $(4, 1)$, a point that already satisfies the equation. If we change the point while maintaining the condition $x_1 + 3x_2 = 7$, we can go to $(1, 2)$ but not to $(2, 2)$, for example.

Suppose now that another condition is imposed, in addition to the first one. Say that we require $2x_1 - 5x_2 = -8$. This in effect restricts our point to a single location, namely $(1, 2)$. The point has lost its degrees of freedom. Each added condition further restricts the point, and we can even make the point *disappear*—meaning that it cannot satisfy all the conditions laid down. In the present example, this occurs if we require further that $x_1 - x_2 = 0$.

Unfortunately, our intuition can be faulty, and that is where the science of linear algebra enters the picture. For example, suppose that we require $x_1 + 3x_2 = 7$, as before, and add the condition $2x_1 + 6x_2 = 14$. This does *not* further restrict the point, because this *new* condition is the same as the first (since it is a simple multiple of the first equation). That is the first *unusual* case. Another unusual case is illustrated by making our second condition $2x_1 + 6x_2 = 8$. Now *no* point satisfies the two conditions. Each equation by itself represents a line, but these two lines are parallel and non-intersecting. Hence, there is no point that satisfies both equations simultaneously.

Next, one can contemplate more complicated situations, such as linear equations with three variables. Again, we can start with an unrestricted point $\mathbf{x} = (x_1, x_2, x_3)$; it has three degrees of freedom. If we impose a single equation, such as $3x_1 - 4x_2 + 5x_3 = 9$, we find that our variable point is now confined to a plane and has only two degrees of freedom. If we add another linear condition, such as $x_1 - 5x_2 - x_3 = 7$, we find that the points satisfying both equations lie on a line. But there are other cases besides this *normal* or *expected* case. Two equations might describe an impossible case (a pair of parallel planes), or they might not actually represent two different conditions. Going on to three equations, one might expect that these three equations

$$\begin{cases} 3x_1 - 4x_2 + 5x_3 = 9 \\ x_1 - 5x_2 - x_3 = 7 \\ 5x_1 - 14x_2 + 3x_3 = 23 \end{cases}$$

would define a single point in $\mathbb{R}^3$, but in fact they describe a plane. Can you see why?

With more variables and more equations the situation becomes more complicated, but we shall develop methods for understanding these problems, no matter how many equations and variables are present. In fact, that is our first major goal: to understand the set of all solutions to any given system of linear equations.

Application: Feeding Bacteria

From time to time in this book, we interrupt the mathematical proceedings to give applications of the theory. Here is such an example.

EXAMPLE 10 A bacteriologist has placed three types of bacteria, labeled B_1, B_2, and B_3, in a culture dish, along with certain quantities of three nutrients, labeled N_1, N_2, and N_3. She knows the amounts of each nutrient that can be consumed by each bacterium in a 24-hour period. These data are collected in a table:

	B_1	B_2	B_3
N_1	4	2	6
N_2	3	1	2
N_3	7	5	2

This table tells us, for example, that each bacterium B_1 in one day can consume 4 units of N_1, 3 units of N_2, and 7 units of N_3. How many bacteria of each type can be supported daily by 4200 units of N_1, 1900 units of N_2, and 4700 units of N_3?

SOLUTION Denote by x_1, x_2, and x_3 the number of bacteria of each type represented in the culture. Considering just the first nutrient and noting how it can be consumed by the three types of bacteria, we have the equation

$4x_1 + 2x_2 + 6x_3 = 4200$. The other two equations governing the nutrients N_2 and N_3 are $3x_1 + x_2 + 2x_3 = 1900$ and $7x_1 + 5x_2 + 2x_3 = 4700$. The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 4 & 2 & 6 & 4200 \\ 3 & 1 & 2 & 1900 \\ 7 & 5 & 2 & 4700 \end{array}\right]$$

The steps in the row reductions are these:

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 3 & 1 & 2 & 1900 \\ 7 & 5 & 2 & 4700 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 0 & -2 & -10 & -5000 \\ 0 & -2 & -26 & -11400 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 0 & 1 & 5 & 2500 \\ 0 & 1 & 13 & 5700 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 0 & 1 & 5 & 2500 \\ 0 & 0 & 8 & 3200 \end{array}\right]$$

The calculation of the solution goes like this:

$$\begin{aligned} x_3 &= 3200/8 = 400 \\ x_2 &= 2500 - 5x_3 = 2500 - 2000 = 500 \\ x_1 &= 2300 - 4x_3 - x_2 = 2300 - 1600 - 500 = 200 \end{aligned}$$

Hence, the solution is $x_1 = 200$, $x_2 = 500$, and $x_3 = 400$ and we obtain $\mathbf{x} = (200, 500, 400)$. ■

Mathematical Software

Throughout the book, we give examples using Matlab, Maple, and Mathematica, which are currently the most popular mathematical software packages. The websites for these software systems are Matlab (`www.mathworks.com`), Maple (`www.maplesoft.com`), and Mathematica (`www.wolfram.com`). There are many other mathematical software systems available. For example, there is a noncommercial system called Octave that is freely redistributable under the terms of the GNU General Public License of the Free Software Foundation (`www.gnu.org/software/octave`). Unfortunately, each of these systems has its own syntax!

A matrix may undergo Gaussian elimination or a variant of it when put into reduced row echelon form. For example, Matlab produces the reduced row echelon form using Gauss–Jordan elimination with *partial pivoting* in which a tolerance parameter is used to test for negligible column elements. Partial pivoting is a process by which pivot elements are selected to reduce roundoff error, and may involve interchanging rows.

EXAMPLE 11 Here is a seven decimal-place example of two linear equations with two variables:

$$\begin{cases} 3.215793x_1 + 82.13459x_2 = 5.332873 \\ 9.300567x_1 - 1.776321x_2 = -12.99334 \end{cases}$$

How can we use the computer to find the solution of this system?

SOLUTION In a computer on which Matlab has been installed, we input the data for our problem by typing this information followed by the commands to solve the system.

```
Matlab
format long
A = [3.215793,82.13459;9.300567,-1.776321]
b = [5.332873;-12.99334]
x = A\b
r = A*x - b
```

In Matlab, spaces can be used in place of the commas between array entries. Matlab does all its computations in full precision but offers various formats for displaying the results. The command `format long` requests answers in full floating point form, with 15 decimal places. The command `format rat` requests answers as ratios of small integers. The command `format short` produces answers with five-digit floating point values; this format is the default. In this example, Matlab responds by printing all the input data (*echoing*) and the solution of the problem $\mathbf{x} \approx (-1.37437016149456,\ 0.11873880352654)$. We then ask for an independent verification of the *answer* by substituting the numerical values of x_1 and x_2 into the two equations and comparing to the prescribed values of b_1 and b_2. In fact, there is a very small discrepancy, which is due to roundoff errors. The difference is $(-0.089, -0.18) \times 10^{-14}$. (In many types of problems, one can verify the proffered solution by substitution or some other independent check.)

The Maple system handles the same problem as follows:

```
Maple
with(LinearAlgebra):
A := Matrix([[3.215793,82.13459],[9.300567,-1.776321]]);
b := Vector([5.332873,-12.99334]);
x := LinearSolve(A,b);
r := A.x - b;
```

Here, we illustrate with a user-friendly package called `LinearAlgebra`, designed explicitly for carrying out linear algebraic computations.

In the Mathematica software package, there are similar commands, as follows:

```
Mathematica
A = {{3.215793,82.13459},{9.300567,-1.776321}}
b = {5.332873,-12.99334}
x = LinearSolve(A,b)
r = A.x - b
```

We have not explained all the notational devices (vectors and matrices) that are being used in this example because at this stage we only want to emphasize that high-quality software is available to solve problems such as this. The user has only to type input values for such programs, and then request the solution. ■

Maple has commands that perform elementary row operations symbolically:

Replacement	$\mathbf{r}_i \leftarrow \mathbf{r}_i + a\,\mathbf{r}_j$	`RowOperation(A,[i,j],a);`
Scale	$\mathbf{r}_i \leftarrow c\,\mathbf{r}_i$	`RowOperation(A,i,c);`
Swap	$\mathbf{r}_i \leftrightarrow \mathbf{r}_j$	`RowOperation(A,[i,j]);`

As an illustration, we solve the linear system $3x_1+2x_2 = 4$ and $9x_1+7x_2 = 17$ from Example 5.

```
Maple
with(LinearAlgebra):
A := Matrix([[3,2,4],[9,7,17]]);
A1 := RowOperation(A,[2,1],-3);
A2 := RowOperation(A1,[1,2],-2);
A3 := RowOperation(A2,1,1/3);
```

The Basic Linear Algebra Subprograms (BLAS) is a collection or "library" of computer routines. It provides standard building blocks for performing basic vector and matrix operations. For the elementary vector operations, the replacement is `_axpy` to suggest the operation $\mathbf{y} \leftarrow a\,\mathbf{x} + \mathbf{y}$ in which a vector $\mathbf{y}$ is replaced by a scalar a times a vector $\mathbf{x}$ plus the original vector $\mathbf{y}$. Also, there are routines called `_swap` for interchanging two vectors ($\mathbf{x} \leftrightarrow \mathbf{y}$) and `_scale` for scaling a vector $\mathbf{x}$ ($\mathbf{x} \leftarrow c\,\mathbf{x}$). The BLAS routines are particularly useful on high-performance computers and have been extended and improved for each new generation of supercomputer and for handling either dense or sparse data.

Algorithm for the Reduced Row Echelon Form

One algorithm for finding the reduced row echelon form of a matrix is presented here. (The steps described are easily translated into a computer program.)

1. Interchange rows if necessary to place all zero rows on the bottom.
2. Identify the leftmost nonzero column. Say it is pivot column j. Interchange rows to bring a nonzero element to the top row and jth column, which is the pivot position. (Computer programs often choose the largest entry in absolute value in an attempt to minimize round-off errors.) Use the row replacement operation to create zeros in all positions in the pivot column below the pivot position.
3. Repeat Steps **1** and **2** on the remaining submatrix until there are no nonzero rows left. (We have found a row echelon form, but it is not unique.)
4. Beginning with the rightmost pivot, working upward and to the left, use row replacement operations to create zeros in all positions in the pivot column above the pivot position. Scale the entry in the pivot row to create a leading 1.
5. Repeat Step **4**, ending with the unique reduced row echelon form of the given matrix.

Steps **1–3** are called the *Gaussian* or *forward* portion of this algorithm, and Steps **4–5** are the *backward* portion. An alternative algorithm called *Gauss-Jordan elimination* combines the forward and backward portions of the algorithm by doing the elimination steps above and below the pivot positions. Although students may find the Gauss-Jordan elimination useful for pencil and paper calculations, it is more computationally intensive for computer programs.

SUMMARY 1.1

- Point–slope form of a line: $y = mx + b$ where m is the slope and b is the intercept
- Two-point form of a line through points (x_0, y_0) and (x_1, y_1): $y - y_0 = m(x - x_0)$ where $m = (y_1 - y_0)/(x_1 - x_0)$
- Lines L_1 and L_2 in $\mathbb{R}^2$: there can be a unique solution (intersect once), no solution (parallel), or infinitely many solutions (co-linear)
- Matrix form of a system of linear equations: $\mathbf{Ax} = \mathbf{b}$
- Augmented matrix: $[\mathbf{A} \mid \mathbf{b}]$
- Gaussian elimination: $\mathbf{Ax} = \mathbf{b}$ becomes (after row reduction) $\mathbf{Ux} = \mathbf{c}$, where $\mathbf{U}$ is an upper triangular matrix
- Row-equivalent augmented matrices: $[\mathbf{A} \mid \mathbf{b}] \sim [\mathbf{U} \mid \mathbf{c}]$
- Elementary row operations:
  - **(replacement)** Add to row i a multiple of row j, where $i \neq j$ $(\mathbf{r}_i \leftarrow \mathbf{r}_i + a\,\mathbf{r}_j)$
  - **(scale)** Multiply row i by c, a nonzero scalar $(\mathbf{r}_i \leftarrow c\,\mathbf{r}_i)$
  - **(swap)** Interchange the two rows i and j $(\mathbf{r}_i \leftrightarrow \mathbf{r}_j)$
- Consistent system has at least one solution; inconsistent system has no solution
- Pivots: leading 1's in reduced row echelon form; pivot positions: locations where the pivots will be at the end of the reduction process
- The reduced row echelon form (unique): zero rows at bottom; pivot rows form a staircase pattern (with possibly different widths of steps); pivot columns contain only the pivot entry 1
- One row echelon form (non-unique): zero rows are moved to the bottom; the rows containing leading nonzero elements form a staircase pattern; columns containing leading nonzero elements contain only 0's elsewhere

KEY CONCEPTS 1.1

Linear equations, systems of linear equations, lines and planes, coefficient matrix, augmented matrix, vector of unknowns, vector of right-hand sides, Gaussian elimination, triangular systems, elementary row operations (replacement, scale, swap), row equivalence of matrices, consistent and inconsistent systems, row-reduction process, reduced row echelon form, pivots and pivot positions, row echelon form, bacteria-nutrition application, using mathematical software, algorithm for reduced row echelon form

GENERAL EXERCISES 1.1

1. Solve this system of equations and verify your answer:

$$\begin{cases} \quad\quad\; 2x_2 - 3x_3 = -11 \\ 4x_1 + x_2 + 3x_3 = 34 \\ \quad\quad\quad\quad\; 5x_3 = 35 \end{cases}$$

2. Solve this system of linear equations and verify your answer:

$$\begin{cases} \quad\quad\quad\quad\quad 3x_1 = 9 \\ 2x_1 - 5x_2 + 6x_3 = -28 \\ \quad -4x_1 + 5x_3 = -32 \end{cases}$$

3. Solve this system by Gaussian elimination and verify your answer:

$$\begin{cases} 2x_1 + 3x_2 = -3 \\ 6x_1 + 4x_2 = -5 \end{cases}$$

4. Solve the system whose augmented matrix is given here, and verify your answer:

$$\left[\begin{array}{cc|c} -3 & 4 & 37 \\ 2 & -5 & -41 \end{array}\right]$$

5. (Continuation.) What are a_{21} and a_{12} in the matrix displayed in the preceding problem?

6. Let $\begin{cases} 3x_2 + 7x_3 + x_1 = 42 \\ x_2 + 2x_1 = 6 \\ 3x_3 + 11x_1 = 76 \end{cases}$

Solve this system of equations by carrying out the row-reduction process to reduced row echelon form.

7. Carry out Gaussian elimination on the system of equations in Example 3 (p. 5). Does that system have any solutions other than $(1, 3, 2)$?

8. Show that

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 7 & 0 \\ 0 & 0 & 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 3 & 15 & 13 & 4 \\ 0 & 2 & 6 & 35 & 28 & 3 \\ 0 & -2 & -6 & 20 & -6 & -3 \end{bmatrix}$$

9. Solve this system in such a way that fractions enter only in the last step:

$$\begin{cases} 3x + y = -5 \\ 2x + 4y = 7 \end{cases}$$

10. Solve the system of equations whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 3 \\ 5 & 4 & 3 & 10 \end{array}\right]$

11. Solve the system of equations whose augmented matrix is $\left[\begin{array}{ccc|c} 3 & 3 & 4 & 11 \\ 2 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \end{array}\right]$

12. Solve this system: $\begin{cases} 3x_1 + 6x_2 + 6x_3 = 21 \\ 2x_1 + 4x_2 + 5x_3 = 16 \\ 2x_1 + 5x_2 + 4x_3 = 17 \end{cases}$

13. Use the theory of linear equations to determine whether the lines described by these three equations have a point in common: $x + 2y = 1,\ 2x - 3y = 9,\ -3x - 2y = -7$.

14. Solve the system of equations having this augmented matrix, without performing any further row operations:

$$\left[\begin{array}{ccc|c} 0 & 0 & 3 & 12 \\ 1 & 2 & 1 & 12 \\ 0 & 2 & 2 & 6 \end{array}\right]$$

15. Show how to solve these three systems all at the same time:

a. $\begin{cases} x + 3y + 2z = 1 \\ 2x + y + z = 0 \\ 4x - y + 3z = 0 \end{cases}$

b. $\begin{cases} x + 3y + 2z = 0 \\ 2x + y + z = 1 \\ 4x - y + 3z = 0 \end{cases}$

c. $\begin{cases} x + 3y + 2z = 0 \\ 2x + y + z = 0 \\ 4x - y + 3z = 1 \end{cases}$

16. Draw graphs of the two lines having equations $x + y = 13$ and $-2x + y = 4$. From the graphs, estimate where the lines intersect. Confirm the estimate by solving the system of two equations. Convert the system of equations to reduced row echelon form.

17. Two lines are given in $\mathbb{R}^2$, namely, $-x + y = 1$ and $2x + y = 4$. Find the point of intersection of these two lines. Then investigate what happens if we change the second equation to $2x - 2y = 4$. Finally, find out what happens if we change the second equation to $2x - 2y = -2$. What do you learn from making small changes to the system?

18. It is claimed that the reduced row echelon form of the matrix

$$\begin{bmatrix} 13 & 17 & -31 & 1097 \\ 11 & -19 & 7 & -413 \\ 5 & 3 & 29 & -359 \end{bmatrix} \text{ is}$$

$$\begin{bmatrix} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & 23 \\ 0 & 0 & 1 & -17 \end{bmatrix}$$

How can we verify or disprove the claim without going through the complete row-reduction process? (Think about systems of equations represented by the two matrices.)

19. Solve these two systems of linear equations and check your work with an independent verification. In each case the augmented matrix is shown:

$$\left[\begin{array}{cc|c} 2 & -1 & a \\ 3 & 4 & b \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & -1 & 11 \\ 3 & 4 & -11 \end{array}\right]$$

20. Find the reduced row echelon forms of these matrices:

a. $\begin{bmatrix} 0 & 3 & 0 & 5 \\ 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 8 & 3 & 1 & 6 \end{bmatrix}$ **b.** $\begin{bmatrix} -12 & 0 & -1 & 2 \\ 16 & 3 & 1 & 0 \\ 20 & 3 & 2 & 4 \\ 12 & 3 & 1 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 1 \\ 4 & 2 \\ -6 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 3 & 4 & -6 \\ 1 & 2 & 1 \end{bmatrix}$

21. In an $n \times n$ matrix whose elements are $a_{ij} = (-1)^{i+j}$, how many positive terms are there? (A formula is sought.)

22. Consider $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -4 & 5 & -23 \\ 2 & 2 & 9 & t \end{array}\right]$

This is a system of equations in which one element t can change. Find values of the parameter t for which we can obtain solutions to this augmented matrix. Explain the implications of this example in the theory of linear equations.

23. Suppose that we have an equation of the form $f(x) = g(x)$, where f and g are real-valued functions of the real variable x. We want to determine x from this equation. Certainly, we can proceed to the equation $[f(x)]^2 = [g(x)]^2$. Give an example to show that solutions of the second equation are not necessarily solutions of the first. Why is the situation here different from the one discussed in this section?

24. By obtaining the reduced row echelon form, find the solution to this pair of equations: $3x_2 + x_1 = 17$ and $2x_1 + 7x_2 = 39$

25. Use the row-reduction techniques to solve this system, in which you may assume $c \neq 0$ and $3c + 5a \neq 0$:

$$\begin{cases} ax + 3y = 7 \\ cx - 5y = -4 \end{cases}$$

26. Explain: If a system of linear equations has exclusively rational numbers for the data a_{ij} and b_i, and if the system has a solution, then it will have a rational solution. (A real number is said to be *rational* if it can be expressed as the quotient of two integers.)

27. Complete all the steps in Example 9 (p. 19) without involving fractions until the last step.

28. Find all the solutions of this nonlinear system of equations: $\begin{cases} 3x^2 - 5y^3 = -123 \\ 7x^2 + 4y^3 = 136 \end{cases}$

29. Let $\begin{cases} 3x^3 - \ln y = 77 \\ 2x^3 + \ln y^3 = 66 \end{cases}$. Find all the solutions of this nonlinear system. (Here $\ln = \log_e$.)

30. **a.** Using the notation $\mathbf{r}_i \leftarrow \alpha\,\mathbf{r}_j + \mathbf{r}_i$, describe the steps used in the row reduction of Example 8 (p. 14) .
b. Use the systematic approach described in the text.

31. Solve the bacteria-nutrition problem when the given data are as in the following table and the nutrients are supplied daily in amounts of 1800 units of N_1, 1500 units of N_2, and 2500 units of N_3. The answer states the number of each type of bacterium to be inserted in the culture.

	B_1	B_2	B_3
N_1	2	1	5
N_2	1	3	1
N_3	3	1	7

32. Consider $\left[\begin{array}{rrr|r} 1 & 3 & 2 & 12 \\ 5 & -7 & 1 & 7 \\ -11 & 33 & t & 23 \end{array}\right]$

Investigate the system of linear equations having this augmented matrix. Here t is a parameter allowed to run over $\mathbb{R}$. What happens if t approaches 5? The moral here is that the solution of a system of linear equations may be a discontinuous function of the data!

33. Explain why this is really a linear equation: $3x(2y+5) + \log(x^2) - 2y(6+3x)$ $= 13 + 2\log x$

34. Explain why this nonlinear system of equations is inconsistent if we allow only real numbers (not complex numbers) as solutions:
$\begin{cases} 3x^2 + 4y^3 = 7 \\ -2x^2 + 3y^3 = 18 \end{cases}$

35. Solve this system of three linear equations: $-5 = x_1 + 2x_3 + 3x_2$, $4x_2 - 4x_3 + 2x_1 = 14$, $x_1 + 2 + x_3 + 3x_2 = 0$

36. Consider $\begin{cases} 7\ln(x^3) + 2\ln(x^2) + y^2 = 77 \\ 2\ln x + 5y^2 = 16 \end{cases}$

Solve this pair of equations for x and y using logarithms to base e.

37. You have seen in General Exercises 28, 29, 33, 34, and 36 examples of nonlinear equations that yield to the techniques of changing variables. Linear changes of variables in linear equations can also be used. Let $\begin{cases} 3x - 2y = 4 \\ 7x + 5y = 21 \end{cases}$. What system of equations results from this system when we change variables like this: $x = u - v$ and $y = u + v$?

38. Let $\begin{cases} 3x_1 - 5x_2 = 17 \\ -x_1 + 2x_2 = 23 \end{cases}$. In this system of equations make the change of variables $x_1 = 2u_1 + 5u_2$ and $x_2 = u_1 + 3u_2$. The new system of equations, involving u_1 and u_2, can be much simpler than the original system of equations. You should find that $x_1 = 149$ and $x_2 = 86$.

39. There exists a matrix in reduced row echelon form such that one column can be removed, leaving a matrix that is also in reduced row echelon form. Find one or more examples of this phenomenon.

40. Consider the matrices $\begin{bmatrix} 3 & 6 & 5 \\ 1 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 4 & 5 & 5 \\ 2 & 7 & 5 \end{bmatrix}$. Are these two matrices row equivalent to each other? Why or why not?

41. Let $\begin{cases} 2x - 3y = -1 \\ 4x + 5y = 53 \end{cases}$.

Solve this system of linear equations by converting the augmented matrix to reduced row echelon form.

42. Consider three planes in $\mathbb{R}^3$ whose equations are $3x + y = 6$, $5x + y + z = 6$, and $3x + z = 1$. Do they have a point in common? If so, find that point (or points).

43. Let $\begin{cases} 2x - 3y = -1 \\ 4x + 5y = 53 \end{cases}$.

Solve this system of linear equations by converting the augmented matrix to reduced row echelon form.

44. Criticize this solution of a system of three linear equations: $-x+y = 1$, $2x+y = 4$, and $3x - y = 2$. To solve this, we can ignore the third equation because the first two equations by themselves give us $x = 1$ and $y = 2$. Hence, the solution is $(1, 2)$.

45. Let $\begin{cases} 2x_1 + 3x_2 - x_3 = -5 \\ 4x_1 - x_2 + 2x_3 = 24 \\ 3x_1 + x_2 - 3x_3 = -8 \end{cases}$.

Solve this system using only integers.

46. (Challenging.) Establish that if a matrix has all integer entries, then it is row equivalent to a matrix in row echelon form having only integer entries. Can we make the same assertion for the reduced row echelon form?

47. If possible, give an example of a 1×5 matrix that is not in row echelon form. Then give an example of a 1×5 matrix that is in row echelon form but not in reduced row echelon form.

48. Which of these matrices is in reduced row echelon form: $[0 \quad 2 \quad 2]$, $[2 \quad 1 \quad 3]$, $[4 \quad 2 \quad 2]$, $[0 \quad 0 \quad 1]$?

49. Consider the matrices $\begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 1 \\ 5 & 7 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 7 & 1 \\ 0 & 17 & 1 \\ 0 & 7 & 1 \end{bmatrix}$. Are these two matrices row equivalent to each other? Why or why not?

50. Let $\begin{cases} 3x - y = 2 \\ 2x + 5y = -4 \\ 7x - 25y = 1 \end{cases}$.

Show that each pair of these equations has a solution but the entire system does not. Interpret geometrically.

51. Consider $\begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}$

Which solution is better?
$\widehat{\mathrm{x}} = (0.341, -0.087)$ or
$\widetilde{\mathrm{x}} = (0.999, -1.001)$. Explain.

TRUE–FALSE EXERCISES 1.1

1. When written out in detail, the expression $\sum_{i=1}^{n} a_{ij}$ looks like this: $a_{i1} + a_{i2} + \cdots + a_{in}$.

2. When written out in detail, the expression $\sum_{j=1}^{n} a_{ji}$ is equivalent to $a_{ni} + \cdots + a_{2i} + a_{1i}$.

3. With our standard notation, the numbers $a_{i1}, a_{i2}, \ldots, a_{in}$ occupy column i in the $n \times n$ matrix **A**.

4. With our standard notation, the numbers $a_{1i}, a_{2i}, \ldots, a_{ni}$ occupy row i in the $n \times n$ matrix $\mathbf{A}$.

5. This matrix is in row echelon form:

$$\begin{bmatrix} 5 & 4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

6. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. None of these four matrices is in reduced row echelon form:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

9. The following matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 7 & 0 & 6 & 8 \\ 0 & 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10. This matrix is in row echelon form:

$$\begin{bmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 2 & 4 \end{bmatrix}$$

11. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 5 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

12. The number of pivots in a matrix is equal to the number of rows that contain pivots.

13. These two matrices are row equivalent to each other:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$

14. A row-equivalent pair of matrices is

$$\begin{bmatrix} 13 & 8 & 5 & 2 \\ 0 & 8 & -7 & 4 \\ 0 & 0 & 17 & 3 \\ 0 & 0 & 0 & 12 \end{bmatrix} \quad \begin{bmatrix} 11 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

15. These two matrices are row equivalent to each other:

$$\begin{bmatrix} 3 & 6 & 5 \\ 1 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 4 & 5 & 5 \\ 2 & 7 & 5 \end{bmatrix}$$

16. A row-equivalent pair of matrices is

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & -8 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

17. A non–row-equivalent pair of matrices is

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

18. Let $\begin{cases} 3x_1 + 2x_2 - 5x_3 = -1 \\ 4x_2 + x_3 = 14 \\ -2x_3 = -4 \end{cases}$

If (x_1, x_2, x_3) is a solution of this system of equations, then $x_2 = 3$.

19. Consider $\begin{cases} x_1 + 3x_2 + 7x_3 = 2 \\ 4x_2 + x_3 = 8x_1 \\ 2x_3 + 2 + 3x_1 = -11 \end{cases}$

The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 4 & 1 & 8 \\ 2 & 2 & 3 & -11 \end{array}\right]$$

20. Let $\begin{cases} 5x^4 + 2\sin y = 16 \\ 2x^4 - 4\sin y = 4 \end{cases}$

By introducing some new variables, we can use the row-reduction process to solve the system.

21. If the elements of a matrix **A** are denoted by a_{ij} or $a_{i,j}$ in the standard way, then $a_{4,7}$ is the fourth element in row seven.

22. A $k \times r$ matrix has k rows and r columns.

23. A $k \times r$ matrix has r rows and k columns.

24. A 7×9 matrix has 9 rows.

25. A 9×7 matrix has 7 columns.

26. Every pair of linear equations in two variables has a solution.

27. Every elementary row operation on a matrix can be reversed by another elementary row operation.

28. A typical linear equation in two variables, x and y, would look like this: $5x + 7y + 3xy = 57$.

29. A linear equation in two variables, u and v, might look like this: $15v - 57u = 3u - v + 99$.

30. Suppose that the pair of equations $E_1(\mathbf{x}) = 0$ and $E_2(\mathbf{x}) = 0$ is transformed into a new pair with a *row operation* that leads to $E_1(\mathbf{x}) = 0$ and $E_2(\mathbf{x}) + \alpha E_1(\mathbf{x}) = 0$. If we want to reverse this operation with a row operation, we must first be sure that the number α is *not* zero.

31. Any 1 occurring in the reduced row echelon form of a matrix is called a *pivot element*.

32. None of these four matrices is in row echelon form:

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

33. There exists a nonzero matrix in reduced row echelon form such that one column can be removed, leaving a matrix in reduced row echelon form.

34. Any pair of linear equations in three variables has a solution.

35. Some pair of linear equations in two variables has no solution.

36. Every equation of the form $ax + by = c$ has at least one solution.

37. If two matrices **A** and **B** are row equivalent to each other, then the two matrices have the same number of rows.

38. For any real number α the operation of multiplying a row by α can be reversed with a suitable row operation.

39. A typical linear equation in two variables, x and y, might look like $(x - 5)(y - 7) = 0$.

40. Sometimes a system of two linear equations with two unknowns will have exactly two solutions.

41. A linear equation in two variables, u and v, might look like this: $15v = 89 - 23u$.

42. Suppose that the pair of equations $E_1(\mathbf{x}) = 0$ and $E_2(\mathbf{x}) = 0$ is transformed into a new pair with a row operation that leads to $E_1(\mathbf{x}) = 0$ and $E_2(\mathbf{x}) + \alpha E_1(\mathbf{x}) = 0$. We can reverse the process with a suitable row operation.

43. If a matrix **A** has $a_{11} \neq 0$, then a_{11} is a pivot position.

44. At least two of these four matrices are in reduced row echelon form:

$$\begin{bmatrix} 1 & 1 \\ 0 & 10 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

45. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 3 & 6 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

46. The following matrix is in row echelon form:

$$\begin{bmatrix} 0 & 1 \\ 3 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

47. This matrix is in row echelon form:

$$\begin{bmatrix} 0 & -9 & 3 & 7 & 6 & 7 \\ 0 & 0 & 0 & 6 & 2 & -5 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

48. Let $\begin{cases} 2x_3 = -3 \\ 5x_1 + 2x_2 - 7x_3 = -1 \\ x_2 + 3x_3 = 14 \end{cases}$

If (x_1, x_2, x_3) is a solution of this system of equations, then $x_1 = 1$.

49. One and only one of these matrices is in reduced row echelon form:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

50. If the elements of a 9×9 matrix $\mathbf{A}$ are denoted by a_{ij} in the standard way, then a_{94} is the ninth element in column four.

51. The following matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 9 & 0 & 6 & 8 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

52. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

53. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 7 & -8 \\ 1 & 2 & -1 \end{bmatrix}$

54. This matrix is in row echelon form:

$$\begin{bmatrix} 0 & 3 & 6 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

55. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

56. A 5×7 matrix has 7 columns and 5 rows.

57. Let $\begin{cases} 3x_1 + 2x_2 - 5x_3 = -1 \\ 4x_2 + x_3 = 14 \\ -2x_3 = -4 \end{cases}$

If (x_1, x_2, x_3) is a solution of this system of equations, then $x_3 = -2$.

58. If the elements of a matrix $\mathbf{A}$ are denoted by a_{ij} in the standard way, then a_{83} is the third element in row eight.

59. These two matrices are row equivalent to each other:

$$\begin{bmatrix} 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}$$

60. There is a row operation on a matrix that can be reversed by exactly the same row operation.

61. This matrix is in row echelon form:

$$\begin{bmatrix} 0 & 1 & 4 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

62. $\begin{bmatrix} 2 & 4 & -2 \\ 6 & 21 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$

63. This matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

64. These two matrices are row equivalent to each other:

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -3 \\ 0 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 3 & 8 \end{bmatrix}$$

65. Elementary row operations on an augmented matrix do not change the solution set of the associated linear system.

66. All nonzero matrices have several row echelon forms.

67. All of these four matrices are in row echelon form:

$$\begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

68. Some matrices have several reduced row echelon forms.

69. A typical nonlinear equation in two variables, x and y, may look like this: $5x^2 + 7y^2 = 75$

MULTIPLE-CHOICE EXERCISES 1.1

Always select the first correct answer.

1. Which of these systems has *no* solution?

a. $\begin{cases} 2x_1 - x_2 = 3 \\ x_1 + x_2 = 1 \end{cases}$ **b.** $\begin{cases} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 6 \end{cases}$

c. $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - 2x_2 = 6 \end{cases}$ **d.** $\begin{cases} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 5 \end{cases}$

e. None of these.

2. Let $\begin{cases} 7y + 3x + 4z = 10 \\ 11x + 2y = 5 + 4z \\ 2z - x = 6 - 3y \end{cases}$

Using the x, y, z ordering of variables, what is the augmented matrix of this system?

a. $\left[\begin{array}{ccc|c} 7 & 3 & 4 & 10 \\ 11 & 2 & 5 & 4 \\ 2 & -1 & 6 & 3 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 7 & 3 & 4 & 10 \\ 11 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 3 & 7 & 4 & 10 \\ 11 & 2 & -4 & 5 \\ -1 & 3 & 2 & 6 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 3 & 7 & 4 & 10 \\ 11 & 2 & 4 & 5 \\ -1 & -3 & 2 & 6 \end{array}\right]$

e. None of these.

3. Let $\mathbf{A} = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & -1 \\ 10 & 8 & 1 \end{bmatrix}$

Which of these matrices is row equivalent to **A**?

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **d.** $\begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

e. None of these.

4. Consider $\left[\begin{array}{ccc|c} 2 & 0 & 6 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

If this is an augmented matrix, the solution set of the linear system is

a. All multiples of $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ **b.** $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$

c. All multiples of $\begin{bmatrix} -6 \\ 6 \\ 0 \end{bmatrix}$

d. All multiples of $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

e. None of these.

5. Consider $\begin{cases} tx_1 - (2t)x_2 = 5 \\ 3x_1 + (6t)x_2 = -7 \end{cases}$

Find all values of the parameter t such that this system does *not* have a solution.

a. 0 **b.** 0, −1 **c.** 6, 0 **d.** 1

e. None of these.

6. If the augmented matrix of a system contains the row $[0 \quad 0 \quad 0 \quad 0 \quad 0 \mid 1]$, we can conclude that the system

a. Has a unique solution.

b. Has many solutions.

c. Has infinitely many solutions.

d. Is inconsistent.

e. None of these.

7. If the augmented matrix of a system of equations has $[0 \quad 0 \quad 0 \quad 0 \quad 0 \mid -1]$ as one of its rows, what conclusion can be drawn about the system?

a. It is inconsistent.

b. It has a unique solution.

c. It has many solutions.

d. It has five unknowns.

e. None of these.

8. What is the solution of the system whose *augmented* matrix is $\left[\begin{array}{cc|c} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 4 \end{array}\right]$?

a. This system has no solution.

b. $(-5, 6)$ **c.** $(-7, 1, 4)$ **d.** $(5, -6)$

e. None of these.

9. Let $\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 2 & -8 & 5 \end{array}\right]$ be an augmented matrix. This system has how many solutions?

a. None. **b.** At least two.

c. At most one. **d.** Exactly one.

e. None of these.

10. Consider the system of equations whose augmented matrix is $\left[\begin{array}{cc|c} 14 & 8 & a \\ 21 & 12 & b \end{array}\right]$. Which assertion is correct?

a. It has a solution for some choices of a and b.

b. It is inconsistent for all choices of a and b.

c. It has a solution for all choices of a and b.

d. It has a solution only in the case $a = 0$.

e. None of these.

11. Define five matrices: $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \mathbf{E} = \begin{bmatrix} 5 & 15 \\ 1 & 3 \end{bmatrix}$

We use the symbol $\sim$ to mean *is row equivalent to*. Which relation is correct?

a. $\mathbf{A} \sim \mathbf{B}$ **b.** $\mathbf{C} \sim \mathbf{D}$ **c.** $\mathbf{E} \sim \mathbf{A}$
d. $\mathbf{E} \sim \mathbf{D}$ **e.** None of these.

12. Which matrix is row equivalent to $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$?

a. $\begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **e.** None of these.

13. Which of these matrices is *not* row equivalent to the other three?

a. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -2 & 5 \\ 0 & -7 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$

e. None of these.

14. Which matrix is in reduced row echelon form?

a. $\begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 5 & 4 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. None of these.

15. Let $\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -6 \end{bmatrix}$

What is the *reduced* row echelon form of **A**?

a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. None of these.

16. Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 2 & 4 & -3 \\ 6 & -2 & 4 & 4 \end{bmatrix}$

What is the reduced row echelon form of **A**?

a. $\begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -4 & -4 & -7 \\ 0 & -20 & -20 & -8 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

e. None of these.

17. Consider this matrix: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 7 \end{bmatrix}$

What is its reduced row echelon form?

a. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

e. None of these.

18. Consider this matrix: $\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ 1 & 4 & 6 & 12 \end{bmatrix}$

What is its reduced row echelon form?

a. $\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

e. None of these.

19. We are pairing row operations with their inverses, using the notation suggested in the text. Which pair is incorrect? (In each case, assume $i \neq j$ and $\alpha \neq 0$.)

a. $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$ and $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$

b. $\mathbf{r}_i \leftarrow \alpha\mathbf{r}_i$ and $\mathbf{r}_i \leftarrow \alpha^{-1}\mathbf{r}_i$

c. $\mathbf{r}_i \leftarrow \alpha\mathbf{r}_j + \mathbf{r}_i$ and $\mathbf{r}_i \leftarrow (1/\alpha)\mathbf{r}_j + \mathbf{r}_i$

d. $\mathbf{r}_i \leftarrow \mathbf{r}_i + \alpha\mathbf{r}_j$ and $\mathbf{r}_i \leftarrow \mathbf{r}_i - \alpha\mathbf{r}_j$.

e. None of these.

20. Which of these systems has a solution?

a. $\begin{cases} 2x_1 - x_2 = 3 \\ x_1 + x_2 = 1 \end{cases}$ **b.** $\begin{cases} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 6 \end{cases}$

c. $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - 2x_2 = 6 \end{cases}$ **d.** $\begin{cases} 2x_1 - x_2 = 3 \\ 4x_1 - 2x_2 = 5 \end{cases}$

e. None of these.

21. Consider $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -6 \end{bmatrix}$

What is a row echelon form of this matrix?

a. $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. None of these.

22. What is the solution of the system whose *augmented* matrix is $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -1 & -1 & -1 \\ 2 & 1 & -4 \end{array}\right]$?

a. This system has no solution.

b. $(-5, 6)$

c. $(-7, 1, 4)$

d. $(5, -6)$

e. None of these.

23. Consider these three equations:

$$\begin{cases} 2x + y + z = 7 \\ 2x + y = 8 \\ -x + z = -4 \end{cases}$$

They define three planes in three-space. Which assertion is correct?

a. This system has no solution.

b. This system has many solutions.

c. This system has one and only one solution, namely $(3, 2, -1)$.

d. $(2, -1, 3)$ is one solution.

e. None of these.

24. Consider this augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{array}\right]$$

Which assertion is correct?

a. The system has many solutions.

b. The system has one and only one solution.

c. $(-1, 2, -2, 1)$ is a solution.

d. $(-3, -1, 1, 0)$ is a solution.

e. None of these.

25. In $\mathbb{R}^3$, the equation $2x + 3y + 4z = 24$ is the equation of a plane. Which point is not on this plane?

a. $(0, 8, 0)$ **b.** $(0, 0, 6)$ **c.** $(12, 0, 0)$

d. $(2, 8, 0)$ **e.** None of these.

26. Consider $\begin{cases} 2x - 3y = 5 \\ 6x - 9y = 7 \end{cases}$. This system has

a. Exactly one solution.
b. Exactly two solutions.
c. No solutions.
d. Infinitely many solutions.
e. None of these.

27. The system $\begin{cases} 2x - 7y = 41 \\ 4x + 3y = -3 \end{cases}$ has

a. Exactly one solution.
b. Exactly two solutions.
c. No solutions.
d. Infinitely many solutions.
e. None of these.

28. Let $\begin{cases} 2x - 3y = 5 \\ 6x - 9y = 15 \end{cases}$. This system has

a. Exactly one solution.
b. Exactly two solutions.
c. No solutions.
d. Infinitely many solutions.
e. None of these.

29. Consider $\begin{cases} x + 3y + 2z = 5 \\ 2x + y + z = 2 \\ 4x - y + 3z = 1 \end{cases}$

This system has
a. One solution.
b. Two solutions.
c. No solutions.
d. Infinitely many solutions.
e. None of these.

30. Let $\begin{cases} x + 3y + 2z = 0 \\ 2x + y + z = 0 \\ 4x - y + 3z = 0 \end{cases}$

This system has
a. Only one solution.
b. Three solutions.
c. No solutions.
d. Infinitely many solutions.
e. None of these.

31. The system $\begin{cases} x + 3y + 2z = 0 \\ 2x + y + z = 0 \end{cases}$ has

a. A unique solution.
b. Infinitely many solutions.
c. No solutions.
d. At least one solution.
e. None of these.

32. What is the point of intersection of the lines described by these equations $4x + 2y = 0$, $x - 2y = 2$, $2x + 6y = -4$?
a. $(2, -4)$ **b.** $(4, 1)$ **c.** $(-2, 0)$
d. There is no point of intersection.
e. None of these.

33. Which matrix is in reduced row echelon form?

a. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 5 & 0 & -7 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

e. None of these.

34. The reduced row echelon form of

$$\begin{bmatrix} 0 & 3 & 0 & 5 \\ 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 8 & 3 & 1 & 6 \end{bmatrix} \text{ is}$$

a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. None of these.

35. Let $\mathbf{A} = \begin{bmatrix} 7 & 3 & 5 & -8 \\ 0 & 2 & 6 & 11 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 \\ 6 & 9 & 8 & 5 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 5 & 4 & 9 & 0 \\ 6 & 3 & 2 & 8 \end{bmatrix}$

For these matrices which of these statements is true?
a. $\mathbf{A} \sim \mathbf{C}$ **b.** $\mathbf{B} \sim \mathbf{0}$ **c.** $\mathbf{A} \sim \mathbf{B}$
d. $\mathbf{B} \sim \mathbf{C}$ **e.** None of these.

36. What is the point of intersection for this pair of equations: $3x_2 + x_1 = 17$ and $2x_1 + 7x_2 = 39$?
a. $(1, -4)$ **b.** $(4, 1)$ **c.** $(2, 5)$
d. There is no point of intersection.
e. None of these.

COMPUTER EXERCISES 1.1

1. Use mathematical software such as Matlab, Maple, or Mathematica to solve one or more of the General Exercises in this section: 8, 12, 15, 20, 22, 31, 45.

2. Use mathematical software to find the reduced row echelon form of a 20×20 matrix containing random integers in $[-20, 20]$.

3. Write a computer program for computing the reduced row echelon form of a given $m \times n$ matrix **A**. In the first version of the program, do not attempt any *pivoting strategy*. In other words, just use the natural ordering of the rows in the matrix as they are given and assume that no zero pivot entries are encountered. (Don't worry about the code not working if division by zero arises.) By the use of comments in the code, indicate the portion of the code that does the *forward elimination phase*, *backward elimination phase*, and *scaling phase*. Test the code on several matrices.

4. (Continuation.) In the second version of the program, generalize the code to handle matrices with zero pivots and zero rows and columns. Test the code on matrices such as this one:

$$\begin{bmatrix} 4 & 12 & 2 & 0 & 16 & 1 & 7 & 26 \\ 6 & 18 & -6 & 0 & 42 & -9 & -57 & -36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -6 & 2 & 0 & -14 & 3 & 19 & 12 \\ 1 & 3 & 3 & 0 & -1 & 2 & 17 & 25 \end{bmatrix}$$

5. (Continuation.) In the third version of the program, modify the general code to use the Basic Linear Algebra Subprograms (BLAS) for carrying out the replacement (`_axpy`), swap (`_swap`), and scale (`_scale`) operations. (As originally proposed by Lawson, Hanson, Kincaid, and Krogh [1979], the BLAS are routines that provide standard building blocks for performing basic vector and matrix operations. They are efficient, portable, and widely available in computer systems. They find common use in the development of high-performance linear algebra software. They can be downloaded at `www.netlib.org/blas`)

6. (Continuation.) In the final version of the program, use the Gauss-Jordan method.

In the Gauss–Jordan algorithm (without pivoting) at the kth major step, the pivot entry in row k is scaled to 1 and multiples of row k are subtracted from *all* the other rows so that all elements above and below the pivot element are 0. In other words, the scaling phase is done first and the forward and backward elimination phases are done together.

7. After seeing page after page of simple numerical examples, usually involving small integers, let's explore a more realistic example. Find the reduced row echelon form of this augmented matrix:

$$\left[\begin{array}{rr|r} 1325.9627 & -23.874191 & 4513.1622 \\ -0.31224877 & 531.26915 & -25492.204 \end{array}\right]$$

1.2 VECTORS AND MATRICES

> "Mathematics is the queen of the sciences.
>
> —KARL FRIEDRICH GAUSS (1777–1855),
> *ONE OF THE GREATEST MATHEMATICIANS OF ALL TIMES*

> The word mathematics comes from the Greek μάθημα (*máthēma*), meaning *science, knowledge, or learning* and μαθηματικός (mathēmatikós) meaning *fond of learning*."
>
> —EN.WIKIPEDIA.ORG

We continue the discussion of vectors and matrices begun in Section 1.1. These concepts play a central role in linear algebra, especially the part of the subject that concerns systems of linear equations.

Vectors

A *vector* is conventionally represented as a vertical column of numbers, such as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 72 \\ -24 \\ 5221 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

Vectors can also be written as horizontal arrays. In some contexts, either form can be used. However, when sums and products of vectors and matrices occur, we must observe certain conventions. For typographical reasons, such as writing vectors in-line and saving space, we often write a vector as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ or as $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$. Here the superscript T means *transpose*. It serves to turn a vertical array into a horizontal one and a horizontal array into a vertical one. The entries in the vector are its *components*.

For a fixed value of n, the set of all vectors having n components is denoted by $\mathbb{R}^n$. Thus, $\mathbb{R}^1$ is just the real numbers $\mathbb{R}$; $\mathbb{R}^2$ is the familiar two-dimensional plane; $\mathbb{R}^3$ is the three-dimensional space of our universe. The remaining cases, $n = 4, 5, \ldots$ do not have familiar geometric interpretations.

The special vector $[0, 0, \ldots, 0]^T$ in $\mathbb{R}^n$ is the zero vector or *origin*. It is also written as $(0, 0, \ldots, 0)$ and is denoted by $\mathbf{0}$. The **addition** of two members of $\mathbb{R}^n$ is effected by the following rule, which is termed **vector addition:**

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

For example, we can write $(1, 3, 7) + (5, -6, 3) = (6, -3, 10)$ or $[1, 3, 7]^T + [5, -6, 3]^T = [6, -3, 10]^T$ or

$$\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 10 \end{bmatrix}$$

We say that the addition of two (or more) vectors is done *componentwise*—that is, *component-by-component*. Notice that the zero vector in $\mathbb{R}^n$ has this property: $\mathbf{x} + \mathbf{0} = \mathbf{x}$, for all $\mathbf{x}$ in $\mathbb{R}^n$.

The geometry for the addition of two vectors in $\mathbb{R}^2$ is shown in Figure 1.3. We form the parallelogram with the given vectors as two sides; their sum is the diagonal vector in the parallelogram. The diagram on the left shows that $(4, 1) + (2, 4) = (6, 5)$, while the one on the right shows that $(-5, 1) + (2, 3) = (-3, 4)$.

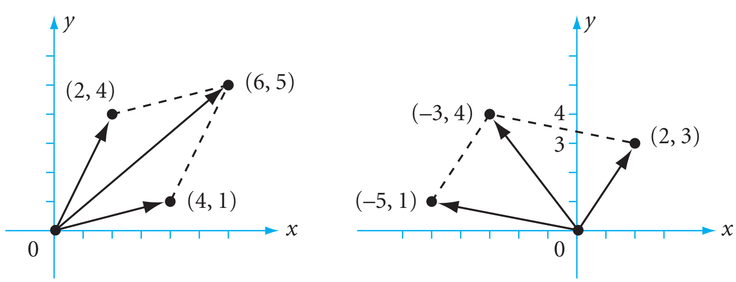

FIGURE 1.3 Addition of pairs of vectors in $\mathbb{R}^2$.

The multiplication of a vector by a **scalar** (i.e., a real number c) is also done component-by-component:

$$c\,\mathbf{x} = c\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

For example, we have

$$4\begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix} = \begin{bmatrix} 12 \\ 28 \\ -20 \end{bmatrix}$$

Figure 1.4 shows scalar multiples of vectors in $\mathbb{R}^2$.

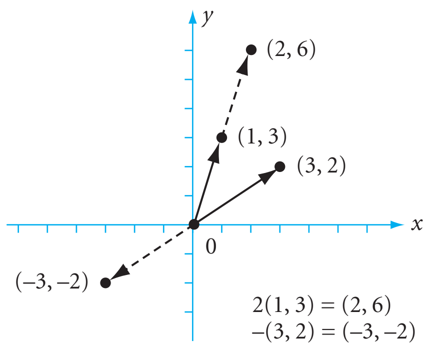

FIGURE 1.4 Scalar multiples of vectors in $\mathbb{R}^2$.

Linear Combinations of Vectors

With these two new definitions, one can form **linear combinations** of vectors, such as

$$3\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} - 5\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 35 \\ -15 \end{bmatrix}$$

These concepts can be put to immediate use in the subject of linear equations. Observe that the system of equations

$$\begin{cases} 3x_1 - 5x_2 + x_3 = 11 \\ 2x_1 + 4x_2 - 3x_3 = -13 \\ 4x_1 - x_2 + 5x_3 = 4 \end{cases}$$

is the same as

$$x_1 \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -13 \\ 4 \end{bmatrix}$$

For another example, consider the vectors $\mathbf{u} = (5, 7)$ and $\mathbf{v} = (-1, 3)$. One can easily calculate a particular linear combination of them such as

$$3\mathbf{u} + 4\mathbf{v} = 3 \begin{bmatrix} 5 \\ 7 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 33 \end{bmatrix}$$

It is more complicated to reverse the direction; that is, if the vector $\mathbf{w} = (11, 33)$ is given, find the linear combination of $\mathbf{u}$ and $\mathbf{v}$ that equals $\mathbf{w}$. We then want to solve the following equation for a and b:

$$a\mathbf{u} + b\mathbf{v} = \mathbf{w}$$

Undertaking this task, we have

$$a \begin{bmatrix} 5 \\ 7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 33 \end{bmatrix} \qquad \text{or}$$

$$\begin{cases} 5a - b = 11 \\ 7a + 3b = 33 \end{cases} \qquad \text{or} \qquad \left[\begin{array}{cc|c} 5 & -1 & 11 \\ 7 & 3 & 33 \end{array}\right]$$

The augmented matrix can be transformed to reduced row echelon form as follows (where we took special care to avoid fractions):

$$\left[\begin{array}{cc|c} 5 & -1 & 11 \\ 7 & 3 & 33 \end{array}\right] \sim \left[\begin{array}{cc|c} 5 & -1 & 11 \\ 2 & 4 & 22 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 11 \\ 5 & -1 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & -11 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

EXAMPLE 1 Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. These are two vectors in $\mathbb{R}^2$. Is every point in $\mathbb{R}^2$ a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$?

SOLUTION *Yes*, because for any $\mathbf{x} \in \mathbb{R}^2$,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

■

Here is a harder problem.

EXAMPLE 2 Is every point of $\mathbb{R}^2$ a linear combination of the vectors $\mathbf{u} = (5, 2)$ and $\mathbf{v} = (7, 3)$?

SOLUTION We try to solve for scalars α and β in the equation

$$\alpha\mathbf{u} + \beta\mathbf{v} = \alpha \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathbf{b}$$

where $\mathbf{b} = (b_1, b_2)$ is an arbitrary vector in $\mathbb{R}^2$. The augmented matrix and its reduced row echelon form are

$$\left[\begin{array}{cc|c} 5 & 7 & b_1 \\ 2 & 3 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 5 & 7 & b_1 \\ 4 & 6 & 2b_2 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 3b_1 - 7b_2 \\ 0 & 1 & -2b_1 + 5b_2 \end{array}\right]$$

The reader can fill in the omitted steps. The answer is *yes*, and we find these values: $\alpha = 3b_1 - 7b_2$ and $\beta = -2b_1 + 5b_2$ for any vector $\mathbf{b} \in \mathbb{R}^2$. ■

EXAMPLE 3 Is the vector $\mathbf{w} = (-1, 3, 7)$ a linear combination of the vectors $\mathbf{u} = (4, 2, 7)$ and $\mathbf{v} = (3, 1, 4)$?

SOLUTION We want to solve the vector equation

$$x \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix} \quad \text{or} \quad \begin{cases} 4x + 3y = -1 \\ 2x + y = 3 \\ 7x + 4y = 7 \end{cases}$$

The augmented matrix and its reduced row echelon form are

$$\left[\begin{array}{cc|c} 4 & 3 & -1 \\ 2 & 1 & 3 \\ 7 & 4 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, we obtain $x = 5$ and $y = -7$. We can verify the results: $5(4, 2, 7) - 7(3, 1, 4) = (-1, 3, 7)$. Most vectors in $\mathbb{R}^3$ are *not* linear combinations of $\mathbf{u} = (4, 2, 7)$ and $\mathbf{v} = (3, 1, 4)$. The vectors that are linear combinations of $\mathbf{u}$ and $\mathbf{v}$ lie on a plane in $\mathbb{R}^3$. Why? ■

Matrix–Vector Products

Now let **A** be an $m \times n$ matrix, thought of as a collection of column vectors. We can write

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$$

Here $\mathbf{a}_j$ denotes the jth column vector in **A**: $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$.

DEFINITION

The **matrix–vector product Ax** *of an $m \times n$ matrix* **A** *and a column vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *is defined to be*

$$\mathbf{A}\,\mathbf{x} = x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\,\mathbf{a}_n$$

Here the scalars x_j are the components of the column vector **x**, *and the column vectors $\mathbf{a}_j$ are the columns of* **A**.

We describe **Ax** as a linear combination of the columns in **A** with coefficients taken to be the components of the vector **x**. Suppose that **A** has dimensions $m \times n$, and let the generic entries in **A** be written as a_{ij}, where $1 \le i \le m$ and $1 \le j \le n$. Using the columns $\mathbf{a}_j$ as above, we write

$$\mathbf{A} = [\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n\ ]$$

By the definition above, we have

$$\mathbf{Ax} = [\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n\ ] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} x_j \mathbf{a}_j$$

To recover one component of **Ax**, we write

$$(\mathbf{Ax})_i = \sum_{j=1}^{n} x_j[\mathbf{a}_j]_i = \sum_{j=1}^{n} x_j a_{ij} = \sum_{j=1}^{n} a_{ij}x_j = \mathbf{r}_i\mathbf{x}$$

This last expression is a genuine matrix product of a $1 \times n$ vector $\mathbf{r}_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$ and an $n \times 1$ vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$. In this situation, we are denoting the ith row of $\mathbf{A}$ by $\mathbf{r}_i$. Thus, to get the ith component of $\mathbf{Ax}$, we compute the vector product

$$\mathbf{r}_i\mathbf{x} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} a_{ij}\, x_j$$

EXAMPLE 4 Express as a single vector the product $\begin{bmatrix} 1 & 5 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

SOLUTION We must carry out a calculation to do this:

$$2\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 7\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + \begin{bmatrix} 35 \\ 7 \\ 28 \end{bmatrix} = \begin{bmatrix} 37 \\ 13 \\ 32 \end{bmatrix}$$

■

EXAMPLE 5 Express as a single vector the product

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

SOLUTION Again, we must carry out the multiplication of a matrix times a vector, as follows:

$$x_1\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \\ 10x_1 + 11x_2 + 12x_3 \end{bmatrix}$$

■

The Span of a Set of Vectors

DEFINITION

The set of all linear combinations of a set of vectors is called the **span** *of that set of vectors.*

Our work in the preceding text shows that the span of the set of columns of a matrix **A** is the set of all vectors **Ax**, where **x** runs over all vectors having the right number of entries. Explicitly, if **A** is $m \times n$, then the span of the set of its columns is

$$\text{Col}(\mathbf{A}) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{\mathbf{Ax} : \mathbf{x} \in \mathbb{R}^n\}$$

The notation $\mathbb{R}^n$ signifies the set of all vectors having n components. We explore this topic more fully in Section 5.1.

The span of the set of columns in a matrix **A** is called the **column space** of **A** and is written Col(**A**). Here are some relevant examples. From Example 1, the span of the set consisting of the two vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ is $\mathbb{R}^2$; that is,

$$\text{Span}\{\mathbf{e}_1, \mathbf{e}_2\} = \text{Col}(\mathbf{I}_2) = \mathbb{R}^2$$

From Example 2, the span of the pair $(5, 2)$ and $(7, 3)$ is $\mathbb{R}^2$ as well.

Obviously, the span of the set of three vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ is $\mathbb{R}^3$; that is,

$$\text{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \text{Col}(\mathbf{I}_3) = \mathbb{R}^3$$

By referring to the calculations already done in Example 3, the span of the pair $(4, 2, 7)$ and $(3, 1, 4)$ contains $(-1, 3, 7)$, but not $(1, 0, 0)$. For the last part of this example, we let

$$x \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We form the augmented matrix and undertake the row reduction:

$$\left[\begin{array}{cc|c} 4 & 3 & 1 \\ 2 & 1 & 0 \\ 7 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The system has no solution and is therefore characterized as *inconsistent*.

The next example is more complicated.

EXAMPLE 6 Give a simple description for the span of $\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = (4, 2, 7)$ and $\mathbf{v} = (3, 1, 4)$. (Ideally, there will be a simple test that can be applied to a vector to determine whether it is or is not in the span of a given set.)

SOLUTION Following Example 2, we form an augmented matrix and carry out the row reduction:

$$\left[\begin{array}{cc|c} \mathbf{u} & \mathbf{v} & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc|c} 4 & 3 & b_1 \\ 2 & 1 & b_2 \\ 7 & 4 & b_3 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|r} 1 & 0 & 4b_2 - b_3 \\ 0 & 1 & b_1 - 2b_2 \\ 0 & 0 & b_1 + 5b_2 - 2b_3 \end{array}\right]$$

For the consistency of this system, we require $b_1 + 5b_2 - 2b_3 = 0$. This condition on vector $\mathbf{b}$ is necessary and sufficient for $\mathbf{b}$ to be in the span of $\mathbf{u}$ and $\mathbf{v}$. The *simple* description asked for could be that the span of $\{\mathbf{u}, \mathbf{v}\}$ consists of all vectors $\mathbf{b}$ whose components satisfy the equation $b_1 + 5b_2 - 2b_3 = 0$. We write

$$\text{Span}\{\mathbf{u}, \mathbf{v}\} = \{\mathbf{b} \in \mathbb{R}^3 : b_1 + 5b_2 - 2b_3 = 0\}$$

Thus most vectors in $\mathbb{R}^3$ are *not* linear combinations of $\mathbf{u} = (4, 2, 7)$ and $\mathbf{v} = (3, 1, 4)$. The vectors that are linear combinations of $\mathbf{u}$ and $\mathbf{v}$ lie on the plane $b_1 + 5b_2 - 2b_3 = 0$ in $\mathbb{R}^3$. ■

EXAMPLE 7 Is the vector $(42, 6, 76)$ in the span of this set of three vectors: $(1, 2, 11)$, $(3, 1, 4)$, $(7, -4, 3)$?

SOLUTION It will be advantageous to think of all these vectors as column vectors. The question is whether a solution exists for this system of linear equations:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 11 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 42 \\ 6 \\ 76 \end{bmatrix}$$

This equation can be written in equivalent forms as explained previously:

$$\begin{cases} x_1 + 3x_2 + 7x_3 = 42 \\ 2x_1 + x_2 - 4x_3 = 6 \\ 11x_1 + 4x_2 + 3x_3 = 76 \end{cases} \quad \text{or} \quad \left[\begin{array}{rrr|r} 1 & 3 & 7 & 42 \\ 2 & 1 & -4 & 6 \\ 11 & 4 & 3 & 76 \end{array}\right]$$

Here we see one numerical vector as a presumed linear combination of three other numerical vectors. Solving the system of equations answers the question of whether the numerical vector on the right is in the span of the three numerical vectors on the left. Only by solving the system can one answer that question. Turning this over to mathematical software, such as Matlab, we find that the answer is *yes*, and that the needed coefficients are $x_1 = 7/2 = 3.5$, $x_2 = 147/19 \approx 7.7368$, and $x_3 = 83/38 \approx 2.1842$. Check: $133(1, 2, 11) + 294(3, 1, 4) + 83(7, -4, 3) = 38(42, 6, 76)$. ■

Interpreting Linear Systems

In general, we can write a system of linear equations as

$$\mathbf{Ax} = \mathbf{b}$$

Usually, we are given the $m \times n$ **coefficient matrix A** and the m-component **righthand-side vector b** and wish to solve for the n-component **unknown vector x**.

One can interpret **b** as a vector or as a matrix having only one column. The important new idea is that a matrix times a vector will be interpreted as a linear combination of the columns of the matrix, and the coefficients in this linear combination are precisely the components of the vector. The conventions of matrix algebra require that we write the vector **x** in the expression **Ax** as a column vector, that is, an $n \times 1$ matrix. (If we wrote **x** as a row vector, the product **Ax** would not be defined.)

Note that for the product **Ax** to be defined, there must be a match between the number of columns in **A** and the number of components in **x**. If the vectors **x** and **b** are interpreted as matrices (of sizes $n \times 1$ and $m \times 1$, respectively), then the matrix equation $\mathbf{Ax} = \mathbf{b}$ can be written in several equivalent forms, as shown on the next page.

Now we have seen various interpretations of a system of linear equations. These interpretations will occur over and over again in our subsequent work. We can use these new concepts to understand a system of equations $\mathbf{Ax} = \mathbf{b}$. If **A** and **b** are given, such a system challenges us to determine whether **b** is in the span of the columns of **A** and, if so, to find the coefficients needed to express **b** as a linear combination of the columns of **A**.

One now has a loftier viewpoint for the problem of solving a system of linear equations. Think of such a problem in the form $\mathbf{Ax} = \mathbf{b}$ and apply the general theory appropriate to such problems. They are conceptually much simpler in this form, and the field is now open to applying the vast armamentarium of matrix theory to such a problem! At this moment, we have discussed only the Gaussian elimination method for solving these systems, but later other methods will be introduced. (Some of these methods are based upon a completely different approach, which allows one to obtain solutions of low precision quickly and solutions of high precision with increasing effort and time.) Many investigators are working on ever-more-efficient ways of solving extremely large systems of equations, often taking advantage of special properties of systems that occur in applications. (If you find this subject interesting, you can devote your life to it and be paid for doing so!)

The following shows the different ways in which we can think of a system of linear equations.

Equivalent Forms of Ax = b

1. The matrix form:

$$\mathbf{Ax} = \mathbf{b}$$

2. As a compact summation:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (1 \le i \le m)$$

3. As linear equations in complete detail:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

4. As a matrix with vectors (arrays) in great detail:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

5. As an augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

6. As a linear combination of the columns of **A**:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

7. As a linear combination of the column vectors of **A**, denoted by $\mathbf{a}_j$:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

Row-Equivalent Systems

The procedure developed in Section 1.1 now operates as follows. Given the matrix **A** and the righthand-side **b**, form the augmented matrix $[\mathbf{A}|\mathbf{b}]$ and carry out the row-reduction process on this matrix. The solutions of the original system and the new system (obtained by applying the row-reduction process) are the same. Here is the formal statement.

THEOREM 1

Let two linear systems of equations be represented by their augmented matrices. If these two augmented matrices are row equivalent to each other, then the solutions of the two systems are identical.

If $[\mathbf{A}\,|\,\mathbf{b}] \sim [\mathbf{B}\,|\,\mathbf{c}]$, then $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\} = \{\mathbf{x} : \mathbf{B}\mathbf{x} = \mathbf{c}\}$, which is Theorem 1 in symbols.

In Section 1.1, the term **row equivalent** was briefly mentioned. We repeat its definition here:

DEFINITION

Two matrices are **row equivalent** *to each other if each can be obtained from the other by applying a sequence of permitted row operations.*

Recall that the permitted row operations are of the following types: *replacement* ($\mathbf{r}_i \leftarrow c\,\mathbf{r}_j + \mathbf{r}_i$), *scale* ($\mathbf{r}_i \leftarrow c\,\mathbf{r}_i$ with $c \neq 0$), and *swap* ($\mathbf{r}_i \leftrightarrow \mathbf{r}_j$). Remember that in the replacement operation $i \neq j$.

Important facts are these: Two matrices are row equivalent to each other if and only if they have the same reduced row echelon form. Every matrix has one and only one reduced row echelon form. (This fact is proved in Section 1.3, Theorem 6, p. 79.)

We can show that these two matrices are row equivalent to each other:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 8 & 1 & 3 \\ 10 & 4 & 4 \end{bmatrix}$$

because they are row equivalent to the same matrix in reduced row echelon form:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/11 \\ 0 & 1 & 1/4 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 8 & 1 & 3 \\ 10 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/11 \\ 0 & 1 & 1/4 \end{bmatrix}$$

EXAMPLE 8 Find all the solutions to the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the matrix in Example 5 (p. 47) and $\mathbf{b}$ is the column vector with entries $[20, 47, 74, 101]^T$.

SOLUTION We form the augmented matrix and undertake the row reduction:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 4 & 5 & 6 & 47 \\ 7 & 8 & 9 & 74 \\ 10 & 11 & 12 & 101 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 3 & 3 & 3 & 27 \\ 6 & 6 & 6 & 54 \\ 9 & 9 & 9 & 81 \end{array}\right]$$

$$\sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Fortunately to enhance our understanding of linear equations, we have here a new phenomenon: It is not clear exactly what the solution of the problem is. The two nonzero rows in this last matrix stand for these two equations:

$$\begin{cases} x_1 - x_3 = -2 \\ x_2 + 2x_3 = 11 \end{cases}$$

Here there are two conditions placed on a vector having three components, and we expect some arbitrariness in the solution. The easiest way to express the set of all solutions is to write the equations in the more suggestive form

$$\begin{cases} x_1 = -2 + x_3 \\ x_2 = 11 - 2x_3 \end{cases}$$

We are at liberty to assign *any* value we please to x_3 and thereby obtain the two other components of $\mathbf{x}$. (The variable x_3 is therefore called a **free variable**.) For example, if $x_3 = 0$, then $x_1 = -2$ and $x_2 = 11$, giving the solution vector $[-2, 11, 0]^T$. Or we can let $x_3 = 7$, from which $\mathbf{x} = [5, -4, 7]^T$. It is more illuminating to write the solution as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 11 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Here s is a **free parameter** standing for the free variable x_3. The reader should study this equation carefully in order to understand where the vectors on the right came from. It is a bit of *sleight of hand* to get from the preceding pair of equations to the single vector equation. (General Exercises 6, 9, 18, 19, 23, 31, 32, 38–40, and others, at the end of this section, give some practice in this art.) This last form of the solution indicates that all the solutions, taken together in $\mathbb{R}^3$, form a line passing through the point $(-2, 11, 0)$ and having the direction vector $(1, -2, 1)$. The variable x_3 now becomes a free parameter s to which we can assign arbitrary values. Each value chosen leads to a solution of the system. The preceding equation gives us the **general solution** of the system of equations. We recommend that solutions to problems such as this be displayed as shown. Usually there will be some constant vectors plus arbitrary multiples of one or more other vectors.

It is customary to treat the variables in their natural order. But in the example just given, we could go counter to this custom and treat x_1 as the free variable. (See General Exercise 19.) Now the general solution would be written

$$\begin{cases} x_3 = 2 + x_1 \\ x_2 = 7 - x_1 \end{cases} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

In this equation, the free parameter t is used in place of x_1. Notice that the same set of solutions can be expressed as the vector $[0, 7, 2]^T$ plus any scalar multiple of $[1, -2, 1]^T$. It is easy to verify that this gives a solution for every choice of the free parameter. The two forms given above describe the same set of solutions. This general solution is the same line in $\mathbb{R}^3$ as given above because it has the same direction vector $(1, -2, 1)$ and goes through the point $(-2, 11, 0)$ as we see by letting $t = -2$. ■

Let us return to the algebraic construct involved in the product $\mathbf{Ax}$, where $\mathbf{A}$ is an $m \times n$ matrix. This expression has been defined previously. From that definition, we obtain immediately

$$\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{Ax}+\mathbf{Ay} \qquad \text{and} \qquad \mathbf{A}(\alpha\mathbf{x}) = \alpha\mathbf{Ax}$$

if α is a scalar. Because of these two properties, the mapping $\mathbf{x} \mapsto \mathbf{Ax}$ is said to be **linear**. (This mapping notation is useful when we want to show the effect of a mapping but do not wish to assign a name to it.) Showing more detail in this calculation, we let $\mathbf{a}_i$ denote column i in $\mathbf{A}$ and then write

$$\mathbf{A}(\mathbf{x}+\mathbf{y}) = \sum_{i=1}^{n}(\mathbf{x}+\mathbf{y})_i\mathbf{a}_i = \sum_{i=1}^{n}(x_i+y_i)\mathbf{a}_i = \sum_{i=1}^{n}x_i\mathbf{a}_i + \sum_{i=1}^{n}y_i\mathbf{a}_i = \mathbf{Ax}+\mathbf{Ay}$$

$$\mathbf{A}(\alpha\mathbf{x}) = \sum_{i=1}^{n}(\alpha\mathbf{x})_i\mathbf{a}_i = \sum_{i=1}^{n}\alpha x_i\mathbf{a}_i = \alpha\sum_{i=1}^{n}x_i\mathbf{a}_i = \alpha\mathbf{Ax}$$

With an induction argument, we obtain

$$\mathbf{A}\left(\sum_{i=1}^{k}\alpha_i\mathbf{u}_i\right) = \sum_{i=1}^{k}\alpha_i\mathbf{Au}_i$$

In this equation, each $\mathbf{u}_i$ is a column vector and the α_i are scalars.

Consistent and Inconsistent Systems

Naturally, when presented with a system, we might ask first whether it is consistent. If it is not, then some more advanced techniques in a later chapter can be invoked to produce an approximate solution to the problem. We recall that when a system of equations has at least one solution, we say that the system is *consistent*; otherwise it is *inconsistent*. Observe that a system of the form $\mathbf{Ax} = \mathbf{0}$ is always consistent because $\mathbf{0}$ is a solution. (This is called the *trivial solution* of a homogeneous system, which we take up in Section 1.3.) No similar remark can be made about the general case $\mathbf{Ax} = \mathbf{b}$, when $\mathbf{b} \neq \mathbf{0}$.

EXAMPLE 9 Here is an example of an inconsistent system of equations. We show the original system and its reduced row echelon form.

$$\left[\begin{array}{rrr|r} 3 & -4 & -8 & 40 \\ 6 & -10 & -26 & 95 \\ 9 & -12 & -24 & 125 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

SOLUTION The second of these augmented matrices certainly exhibits the inconsistency, because the third equation reads $0x_1 + 0x_2 + 0x_3 = 1$, and this cannot be true. Because the second augmented matrix is the reduced row echelon form of the first, we conclude that the first system is inconsistent, although that fact is *not obvious* from the original matrix. But the row reduction process reveals the inconsistency concealed in the system. ■

The example just given illustrates the following theorem.

THEOREM 2

A system of linear equations, $\mathbf{Ax} = \mathbf{b}$, *is consistent if and only if the vector* $\mathbf{b}$ *is in the span of the set of columns of* $\mathbf{A}$.

PROOF Suppose that $\mathbf{x}$ is a vector such that $\mathbf{Ax} = \mathbf{b}$. When this is written in detail as a linear combination of columns of $\mathbf{A}$ equaling the vector $\mathbf{b}$, we see that $\mathbf{b}$ is indeed a linear combination of the columns of $\mathbf{A}$. The converse is true: if $\mathbf{b} = \sum_{i=1}^{n} x_i \mathbf{a}_i$ where the $\mathbf{a}_i$ are the columns of $\mathbf{A}$ and the x_i are scalars, then the vector $\mathbf{x}$ having components x_i is a solution of the system $\mathbf{Ax} = \mathbf{b}$. ■

THEOREM 3

Let $\mathbf{A}$ *be an* $m \times n$ *matrix. The system of equations* $\mathbf{Ax} = \mathbf{b}$ *is consistent for all* $\mathbf{b}$ *in* $\mathbb{R}^m$ *if and only if the columns of* $\mathbf{A}$ *span* $\mathbb{R}^m$. *In other words,* $\mathrm{Col}(\mathbf{A}) = \mathbb{R}^m$.

PROOF By Theorem 2, consistency of the system for all $\mathbf{b}$ means that every $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of columns of $\mathbf{A}$. In other words, the columns of $\mathbf{A}$ span $\mathbb{R}^m$. ■

THEOREM 4

Let $\mathbf{A}$ *be an* $m \times n$ *matrix. The system of equations* $\mathbf{Ax} = \mathbf{b}$ *is consistent for all* $\mathbf{b}$ *in* $\mathbb{R}^m$ *if and only if each row of the coefficient matrix* $\mathbf{A}$ *has a pivot position.*

PROOF Asserting that each row of the coefficient matrix **A** has a pivot position is equivalent to asserting that the reduced row echelon form of **A** has a pivot in each row. That, in turn, is equivalent to asserting that the columns of **A** span $\mathbb{R}^m$. By Theorem 3, this is equivalent to the system of equations $\mathbf{Ax} = \mathbf{b}$ being consistent for all $\mathbf{b}$ in $\mathbb{R}^m$. ■

THEOREM 5

A system of linear equations is inconsistent if and only if its augmented matrix has a pivot position in the last column.

PROOF Recall that the last column in the augmented matrix is the right-hand-side vector. If the reduced row echelon form of the augmented matrix has a pivot in the last column, the equation represented by the row containing that pivot is inconsistent, because the coefficients of all the variables are 0, whereas the righthand side has a 1. (It has the form $0x_1 + 0x_2 + \cdots + 0x_n = 1$, and no solution is possible.) Conversely, if there is no pivot element in the last column, values can be assigned to all the variables, creating a solution. For example, assign arbitrary values to all the free variables (if there are any) and use the reduced row echelon form to find the values of all the other variables. ■

As mentioned previously, every matrix has a unique reduced row echelon form. It is proven in Theorem 6 in Section 1.3, p. 79. (It's not easy!) This special relationship of **row equivalence** is denoted in this book by the symbol $\sim$. For example, we know that the two following matrices, **A** and **B**, are row equivalent to each other because each is row equivalent to $\mathbf{I}_3$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 10 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3$$

$$\mathbf{B} = \begin{bmatrix} 17 & 31 & -11 \\ 2 & 5 & 47 \\ -19 & 3 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3$$

This example shows that it may be easier to prove $\mathbf{A} \sim \mathbf{I}$ and $\mathbf{B} \sim \mathbf{I}$ than to find a chain of row operations that go directly from **A** to **B**.

Caution

Theorem 5 involves the reduced row echelon form of an augmented matrix $[\mathbf{A} \mid \mathbf{b}]$, whereas Theorem 4 involves only the coefficient matrix **A**. The system of equations $\mathbf{Ax} = \mathbf{b}$ may or may not be consistent for all $\mathbf{b}$ in $\mathbb{R}^m$. In Example 2, the system was found to be consistent for all righthand sides **b:**

$$\left[\begin{array}{cc|c} 5 & 7 & b_1 \\ 2 & 3 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3b_1 - 7b_2 \\ 0 & 1 & -2b_1 + 5b_2 \end{array}\right]$$

The coefficient matrix has a pivot position in each row.

The system in Example 6 can be consistent or inconsistent, depending on the numbers b_1, b_2, b_3. The row reduction yields

$$\left[\begin{array}{cc|c} 4 & 3 & b_1 \\ 2 & 1 & b_2 \\ 7 & 4 & b_3 \end{array}\right] \sim \left[\begin{array}{cc|r} 1 & 0 & 4b_2 - b_3 \\ 0 & 1 & b_1 - 2b_2 \\ 0 & 0 & b_1 + 5b_2 - 2b_3 \end{array}\right]$$

This system is consistent only when $b_1 + 5b_2 - 2b_3 = 0$. When that condition is met, there will *not* be a pivot position in the last column of the augmented matrix.

THEOREM 6

A system of linear equations is consistent if and only if the reduced row echelon form of its augmented matrix does not have a pivot position in the last column.

PROOF This is really just a restatement of Theorem 5. ■

Application: Linear Ordinary Differential Equations

A system of linear ordinary differential equations can be expressed using matrices and vectors. For example, consider the two equations

$$dx/dt = y \qquad dy/dt = -x$$

with initial values $x(0) = 0$ and $y(0) = 1$. Letting

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}$$

we obtain

$$d\mathbf{z}/dt = \mathbf{Az}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with initial conditions

$$\mathbf{z}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The solution is $x(t) = \sin(t)$ and $y(t) = \cos(t)$. When we plot the solutions we obtain two overlapping sine and cosine curves. We can generalize this concept to handle n linear ordinary differential equations by using a vector of size n and a matrix of size $n \times n$.

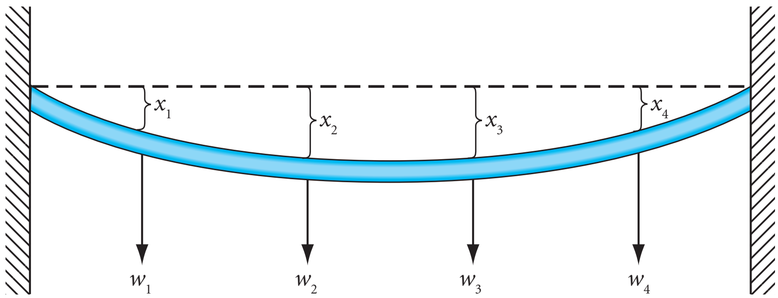

FIGURE 1.5 Bending beam.

Application: Bending of a Beam

Studying the elasticity of building materials can bring in problems of linear algebra via Hooke's law, which has a linear nature. Consider a flexible steel beam supported by posts at each end. If a downward force is applied to the beam somewhere between the supports there will be a deflection in the beam. If there are other such forces (called *stresses*), then we would like to compute the expected deflections due to all the stresses. The deflections are called *strains.*

For example, we consider four forces (stresses) w_1, w_2, w_3, w_4 applied to the beam at four different locations. The beam then suffers deflections (strains) x_1, x_2, x_3, x_4 as shown in Figure 1.5. Define the vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$ and the vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$. In 1676, Robert Hooke noted that in an elastic material strain is proportional to stress.[4] By **Hooke's Law**, there is a linear relationship between the forces and the deflections, given by the equation

$$\mathbf{x} = \mathbf{F}\mathbf{w}$$

[4] Robert Hooke (1635—1703) was at his best when his mind was jumping from one thing to another and not doing just one thing at a time. Throughout his life, Hooke had bitter disputes with fellow scientists, such as claiming that Newton stole some of his own ideas about the theory of light. Consequently, Newton removed all references to Hooke from *The Principia.*

where the matrix **F** is the **flexibility matrix**. The entries in the matrix **F** have to be determined by careful experiments.

In this example, **F** is a 4×4 matrix. The equation $\mathbf{x} = \mathbf{Fw}$ can be written in full detail as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$= w_1 \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \end{bmatrix} + w_2 \begin{bmatrix} f_{12} \\ f_{22} \\ f_{32} \\ f_{42} \end{bmatrix} + w_3 \begin{bmatrix} f_{13} \\ f_{23} \\ f_{33} \\ f_{43} \end{bmatrix} + w_4 \begin{bmatrix} f_{14} \\ f_{24} \\ f_{34} \\ f_{44} \end{bmatrix}$$

$$= w_1\mathbf{f}_1 + w_2\mathbf{f}_2 + w_3\mathbf{f}_3 + w_4\mathbf{f}_4$$

The first column $\mathbf{f}_1$ in the matrix **F** is obtained by measuring the strains that arise from applying a unit stress on the beam at the first point in the diagram. In the same way, the three other columns of **F** are found by placing a unit weight at the three other locations on the beam and measuring the strains. Another way to explain this is to imagine vector **w** to be $\mathbf{e}_1 = (1, 0, 0, 0)$. Physically it means that a unit weight has been placed at the first point on the elastic beam. The deflections are measured at all four points of the beam and entered as column 1 in the matrix **F**. The first column of **F** gives the deflections due solely to a unit force applied at the first point. Similar interpretations are valid for the other columns of **F**. For example, if the flexibility matrix is

$$\mathbf{F} = \begin{bmatrix} 0.2 & 0.1 & 0.3 & 0.4 \\ 0.7 & 0.5 & 0.6 & 0.8 \\ 0.9 & 0.2 & 0.2 & 0.5 \\ 0.6 & 0.3 & 0.4 & 0.9 \end{bmatrix}$$

and the forces are given by $\mathbf{w} = [25, 45, 35, 55]^T$ in millimeters per Newton, then the deflection vector would be $\mathbf{x} = [42, 105, 66, 92]^T$ in millimeters measured from the original four points on the unbent beam.

Mathematical Software

One can use sophisticated mathematical software to carry out tedious calculations. To illustrate, consider the augmented matrix in Example 8 (p. 53). Here are the Matlab commands needed to enter the matrix and to invoke the algorithm for reduced row echelon form:

Matlab
`A = [1,2,3,20;4,5,6,47;7,8,9,74;10,11,12,101]` `rref(A)`

In Matlab, if you wish to see the successive steps in the row-reduction process, replace the command `rref` by `rrefmovie`. Similarly, the commands in Maple are

Maple

`with(LinearAlgebra):`
`A := Matrix([[1,2,3,20],[4,5,6,47],[7,8,9,74],[10,11,12,101]]);`
`ReducedRowEchelonForm(A);`

Finally, the Mathematica commands are

Mathematica
`A = {{1,2,3,20},{4,5,6,47},{7,8,9,74},{10,11,12,101}` `RowReduced[A]`

The Maple software package supports symbolic calculations and can carry out the calculations for the general solution.

Maple

`with(LinearAlgebra):`
`A := Matrix([[1,2,3,20],[4,5,6,47],[7,8,9,74],[10,11,12,101]]);`
`B := ReducedRowEchelonForm(A);`
`C := BackwardSubstitute(B,free='x');`
`Sol := LinearSolve(A,free='x');`

One can also use Mathematica to solve systems with symbols in them such as this one with an arbitrary righthand side:

$$\begin{cases} x + 2y = a \\ 3x + 7y = b \end{cases}$$

Maple	Mathematica
`with(LinearAlgebra):` `solve({x +2*y=a,3*y+7*y=b},{x,y});`	`Solve[{x + 2y == a,` `3x + 7y == b},{x,y}]`

We find $x = 7a - 2b$ and $y = -3a + b$. Some of the Maple symbolic manipulations can also be carried out in Matlab, using its *Symbolic Math Toolbox.*

SUMMARY 1.2

- Vectors in $\mathbb{R}^n$: $\mathbf{x} = (x_1, x_2, \ldots, x_n) = [x_1, x_2, \ldots, x_n]^T$; zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$
- Vector addition: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$; scalar multiplication: $c\mathbf{x} = (cx_1, cx_2, \ldots, cx_n)$
- Elementary unit vectors: $\mathbb{R}^2$: $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$; $\mathbb{R}^3$: $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$
- Matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]_{m\times n}$ where $\mathbf{a}_j = [a_{1j}, a_{2j}, \ldots, a_{nj}]^T$ and is the jth column vector in $\mathbf{A}$
- Matrix–vector product: $\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$, a linear combination of columns in $\mathbf{A}$; $(\mathbf{Ax})_i = \mathbf{r}_i\mathbf{x}$, where $\mathbf{r}_i$ is the ith row of $\mathbf{A}$
- $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is the set of all linear combinations of $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$; $\text{Col}(\mathbf{A}) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$
- Equivalent forms of linear systems:
  - Matrix form: $\mathbf{Ax} = \mathbf{b}$
  - Compact summation: $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for $1 \le i \le m$
  - In complete detail:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

  - As a matrix with vectors:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

  - As an augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

  - As a linear combination of the columns of $\mathbf{A}$:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

  - $\mathbf{Ax} = \mathbf{b}$ is the same as $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$, which is a linear combination of the column vectors $\mathbf{a}_j$ in $\mathbf{A}$
- The mapping $\mathbf{x} \to \mathbf{Ax}$ is linear: $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$ and $\mathbf{A}(c\mathbf{x}) = c\mathbf{Ax}$
- The equation $\mathbf{Ax} = \mathbf{b}$ is consistent if there exists at least one solution; otherwise it is inconsistent
- Theorems:
  - The system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{Col}(\mathbf{A})$.
  - Let $\mathbf{A}$ be an $m \times n$ matrix. The system $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^n$ if and only if $\text{Col}(\mathbf{A}) = \mathbb{R}^m$.
  - Let $\mathbf{A}$ be an $m \times n$ matrix. The system $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^n$ if and only if there is a pivot position in each row of $\mathbf{A}$. Equivalently, $\mathbf{A}$ has m pivot positions.
  - The system $\mathbf{Ax} = \mathbf{b}$ is inconsistent if and only if there is a pivot in the last column of the row-reduced augmented matrix $[\mathbf{A} \mid \mathbf{b}]$.

KEY CONCEPTS 1.2

Vectors, components, transpose of a vector, linear combinations of vectors, matrix–vector product **Ax**, span of a set of vectors, various forms of linear systems, row-equivalent matrices, free parameters and variables, general solution of a system of linear equations, consistent and inconsistent systems of equations, bending beam application, using mathematical software

GENERAL EXERCISES 1.2

1. Solve these two systems: $\begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 \\ 1 & 4 \\ 2 & 1.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ 1 \end{bmatrix}$

What conclusion can be drawn?

2. Let $\mathbf{A} = \begin{bmatrix} 3 & 7 & -4 \\ 5 & -2 & 6 \\ 2 & 1 & -1 \\ 4 & 1 & 2 \end{bmatrix}$ and let **b** be a vector in $\mathbb{R}^4$ such that the system $\mathbf{Ax} = \mathbf{b}$ has a solution. Explain why it has only one.

3. (Continuation.) Let **A** be as in General Exercise 2, and let $\mathbf{b} = [68, -32, 15, 4]^T$ and $\mathbf{x} = [2, 6, -5]^T$. The superscript T indicates that these vectors are to be considered as *column vectors*. Determine whether **x** is a solution of the system $\mathbf{Ax} = \mathbf{b}$.

4. Find a vector that solves the first of these two systems but not the second. Then find a vector that satisfies the second but not the first.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 7 \end{array}\right]$$

5. Without doing any calculations, explain why these two matrices are row equivalent to each other:

$$\begin{bmatrix} 7 & 3 & 5 & -8 \\ 0 & 2 & 6 & 11 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Let $\begin{cases} x - y + z = 4 \\ 2x + y - 3z = 5 \\ -y + 7x - 3z = 22 \end{cases}$

Find all the solutions of the system.

7. Consider $\begin{bmatrix} 1 & 3 & 0 & 5 & 6 \\ 2 & 6 & 1 & 8 & 14 \\ 3 & 9 & 0 & 15 & 18 \end{bmatrix}$

Find the reduced row echelon form of this matrix.

8. Solve the system whose augmented matrix is the following, by finding the reduced row echelon form of the matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & -8 \\ 2 & -3 & 1 & 27 \\ 5 & 2 & 3 & 23 \end{array}\right]$$

9. Find the general solution of this system of equations:

$$\begin{cases} 4x_1 + 12x_2 + 2x_3 + 16x_4 + x_5 + 7x_6 = 26 \\ -2x_1 - 6x_2 + 2x_3 - 14x_4 + 3x_5 + 19x_6 = 12 \\ x_1 + 3x_2 + 3x_3 - 1x_4 + 2x_5 + 17x_6 = 25 \end{cases}$$

10. Show that this system is inconsistent:

$$\begin{cases} x_1 + 3x_2 + 15x_3 = 28 \\ 2x_1 + 4x_2 + 20x_3 = 40 \\ x_1 + 2x_2 + 10x_3 = 27 \end{cases}$$

11. Solve the system of equations whose augmented matrix is $\left[\begin{array}{cccc|c} 2 & 0 & 2 & 8 & 11 \\ 7 & 1 & 2 & 8 & 35 \\ 2 & 0 & 1 & 4 & 9 \\ 1 & 0 & 2 & 8 & 7 \end{array}\right]$

12. Accept the hypothesis that

$$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 4 & 11 & 13 & -1 \\ 2 & 3 & 7 & -8 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 5 & 2 & 6 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Find the solution of the following system:

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 1 \\ 4x_1 + 11x_2 + 13x_3 = -1 \\ 2x_1 + 3x_2 + 7x_3 = -8 \end{cases}$$

13. What are the inverses of these four row operations: $\mathbf{r}_p \leftarrow \mathbf{r}_p + \alpha\mathbf{r}_q$, $\mathbf{r}_q \leftarrow \mathbf{r}_p + \alpha\mathbf{r}_q$, $\mathbf{r}_p \leftrightarrow \mathbf{r}_q$, and $\mathbf{r}_q \leftarrow \beta\mathbf{r}_q$, where $\alpha \neq 0$ and $\beta \neq 0$. In which cases must we assume $q \neq p$ or another hypothesis?

14. Describe the span of the set of columns in the matrix $\begin{bmatrix} 3 & 1 & 4 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

15. Without any calculations, explain why these two matrices are not row equivalent to each other:

$$\begin{bmatrix} 3.2 & 1.3 & 5.5 & 16.1 & 22.8 & 19.2 & 11.5 \\ 1.3 & 3.7 & 2.0 & 19.2 & 11.7 & 16.9 & 12.3 \\ 4.7 & 5.9 & 9.3 & 12.4 & 13.2 & 15.8 & 18.7 \end{bmatrix}$$

$$\begin{bmatrix} 18.5 & 5.8 & 7.7 & 3. & 2.9 & 4. & 11.5 & 8.1 \\ 11.7 & 8.3 & 2.4 & 6. & 1.4 & 2. & 21.3 & 9.8 \\ 37.2 & 9.1 & 5.6 & 3. & 8.2 & 5. & 23.3 & 1.8 \end{bmatrix}$$

16. Sometimes, in applying the row-reduction process, there are zeros already present in the matrix, and one is tempted to take advantage of that fact. However, this may not be possible, and, if so, those zeros will be *sacrificed* in the reduction process. Here is such an example: $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ -3 & -2 & 0 & -4 \\ 5 & 0 & 4 & 3 \end{array}\right]$. Taking the given matrix to be the augmented matrix for a system of equations, find the reduced row echelon form and the solution vector.

17. What test can you devise to ascertain that two equations $ax + by = c$ and $rx + sy = t$ define the same line? (Assume that the coefficients a, b, r, s are all nonzero.)

18. Without performing any row operations, solve the system whose augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & 0 & 6 \\ 5 & -4 & 0 & 1 & 7 \end{array}\right]$$

19. Redo Example 8 (p. 53), using x_1 as the free variable.

20. Let $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (1, 1, 0)$. Test the following four vectors to see which ones are in the span of $\{\mathbf{u}, \mathbf{v}\}$: $\mathbf{w} = (1, -1, 2)$, $\mathbf{x} = (4, 3, 1)$, $\mathbf{y} = (1, 1, 1)$, and $\mathbf{z} = (1, 2, -1)$. Can you devise a simple test for this task, keeping the vectors $\mathbf{u}$ and $\mathbf{v}$ as they are? (The test should be easy to apply to any vector $\mathbf{b}$ in $\mathbb{R}^3$.)

21. Let $\begin{cases} \ln x^{25} + y^2 = 77 \\ \ln x^2 + 5y^2 = 16 \end{cases}$

Solve this system for x and y. The logarithms are based on $e = 2.71828\ldots$.

22. In this problem, we describe matrices by listing their columns, which are vectors in $\mathbb{R}^m$. Explain why if $[\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n] \sim [\mathbf{b}_1\ \mathbf{b}_2 \cdots \mathbf{b}_n]$ and $k < n$, then $[\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_k] \sim [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_k]$. If this turns out to be false, provide a suitable example.

23. Consider

$$\begin{cases} 0x_1 - 4x_2 + 9x_3 + 29x_4 + 14x_5 = 9 \\ 0x_1 - 2x_2 + 3x_3 + 12x_4 + 6x_5 = 2 \\ 0x_1 + 6x_2 - 12x_3 - 41x_4 - 20x_5 = -11 \end{cases}$$

Solve the system. Be sure to identify the free variables (*parameters*) in the general solution. Express the general solution in the manner recommended in the text. (See Example 8 on p. 53.)

24. Explain why the span of the columns in an $m \times n$ matrix $\mathbf{A}$ is identical to the set $\{\mathbf{Ax} : \mathbf{x} \in \mathbb{R}^n\}$.

25. Suppose that a system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent and that a set of coefficients c_i has the property $\sum_{i=1}^m c_i a_{ij} = 0$ for $j = 1, 2, \ldots, n$. Explain why $\sum_{i=1}^m c_i b_i = 0$. Here we have supposed that $\mathbf{A}$ is an $m \times n$ matrix.

26. Determine whether the vector $\begin{bmatrix} -9 \\ 10 \\ 8 \end{bmatrix}$ is in the column space of the matrix

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 0 & 4 \\ 3 & 1 & 3 \end{bmatrix}$$

27. Give an argument why if one matrix can be obtained from another via allowable row operations, then the two matrices have the same reduced row echelon form.

28. If a system of equations $\mathbf{Ax} = \mathbf{b}$ is inconsistent, can we always restore consistency by changing one entry in vector $\mathbf{b}$?

29. Explain why or provide counterexamples: For a pair of vectors $\mathbf{x}, \mathbf{y}$ interpreted as $n \times 1$ matrices:

a. $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$ **b.** $\mathbf{x}\mathbf{y}^T = \mathbf{y}\mathbf{x}^T$

30. Let $\begin{cases} 3x_1 + x_2 + x_3 = a \\ -3x_1 + 9x_2 - 5x_3 = b \\ 6x_1 - 3x_2 + 4x_3 = c \end{cases}$

Find the exact condition on (a, b, c) so that this system is consistent.

31. Describe the solutions of the system whose augmented matrix is $\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 7 & 5 \end{array}\right]$

Indicate which variables are independent (free) and which are dependent.

32. Consider $\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 0 & 6 & 8 \\ 0 & 2 & 0 & 3 & 9 & 7 \end{array}\right]$

Describe the set of all solutions of the system having this augmented matrix. Indicate which variables are independent (free) and which are dependent.

33. Establish that if the matrix $\mathbf{A}$ having entries a_{ij} is in row echelon form, then $a_{ij} = 0$ when $j < i$.

34. By using row operations, determine whether these two matrices are row equivalent to each other:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 9 & 2 \\ 2 & 6 & 1 \\ 2 & -3 & -1 \end{bmatrix}$$

35. Determine whether these two matrices are row equivalent to each other:

$$\begin{bmatrix} 2 & 6 & 4 \\ 1 & 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & 10 \\ 1 & 2 & 10 \end{bmatrix}$$

36. Suppose that the reduced row echelon form of $[\mathbf{A} \mid \mathbf{b}]$ has a pivot in the last column. Explain why the system of equations $\mathbf{Ax} = \mathbf{b}$

is inconsistent, that is, has *no* solution. Is this true for any row echelon form of the augmented matrix?

37. In an $m \times n$ matrix whose elements are $a_{ij} = (-1)^{i+j}$, how many positive terms are there?

38. Find two or more solutions to the system of equations whose augmented matrix is given here, and verify your answer:

$$\left[\begin{array}{ccc|c} 2 & 3 & -5 & 20 \\ 3 & -4 & 6 & -15 \end{array}\right]$$

39. (Continuation.) Find the solution of the system in the preceding exercise that minimizes the expression $x_1^2 + x_2^2 + x_3^2$.

40. Find all solutions of the system whose augmented matrix is $\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 6 & -4 & 1 & 0 \\ 10 & 2 & 11 & 0 \end{array}\right]$

41. Let **A** and **B** be $m \times n$ matrices. Explain why $\mathbf{A} = \mathbf{B}$ if and only if $\mathbf{Ax} = \mathbf{Bx}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Half of this (the *only if*) part is rather obvious. It is the *if* part that requires an idea!

42. For linear systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{By} = \mathbf{c}$, does $[\mathbf{A} \,|\, \mathbf{b}] \sim [\mathbf{B} \,|\, \mathbf{c}]$ imply $\mathbf{x} = \mathbf{y}$? Explain why or why not.

43. Fill in the missing steps in Example 2.

TRUE–FALSE EXERCISES 1.2

1. Consider a system of equations having this augmented matrix: $\left[\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

In this system of equations, the basic variables are x_1 and x_2, whereas the variable x_3 is a *free variable.*

2. (Continuation.) The system referred to in the preceding question has a unique solution, namely $x_1 = 3$, $x_2 = 4$, and $x_3 = 0$.

3. This equation is correct:

$$\begin{bmatrix} 1 & 5 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 37 \\ 13 \\ 34 \end{bmatrix}$$

4. If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & -4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$, then $\mathbf{Ax} = \begin{bmatrix} 4 \\ 58 \end{bmatrix}$

5. The product **Ax**, where **A** is an $m \times n$ matrix and **x** is a column vector with n components, is defined to be the linear combination of the columns of **A** with coefficients equal to the components in the **x** vector.

6. The product **Ax** of a matrix **A** and a vector **x** is defined to be a linear combination of the rows of **A**, with coefficients equal to the components of the vector **x**.

7. The product **Ax** is defined if the number of components in the vector **x** equals the number of rows of **A**.

8. Let **A** be an $m \times n$ matrix in which $m > n$. Then there will exist a vector **b** such that the system of linear equations $\mathbf{Ax} = \mathbf{b}$ is inconsistent.

9. Let **A** be an $m \times n$ matrix for which $n > m$. If the equation $\mathbf{Ax} = \mathbf{b}$ is consistent, then it has many solutions.

10. If $\mathbf{A}$ is a $k \times q$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^k$, then $\mathbf{Ax}$ is in $\mathbb{R}^q$.

11. The matrix equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is $m \times n$, means that $\sum_{p=1}^{n} a_{pq}x_p = b_q$ for $1 \leq q \leq m$. (Here, a_{pq} are elements of the matrix $\mathbf{A}$ and b_q are the components of the vector $\mathbf{b}$.)

12. For any system of linear equations, if there is a free variable, then there will be infinitely many solutions.

13. If a system of linear equations $\mathbf{Ax} = \mathbf{b}$ is consistent, then the vector $\mathbf{b}$ is in the span of the set of columns of $\mathbf{A}$.

14. If the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is in the span of the set of rows of $\mathbf{A}$.

15. A system of linear equations is consistent if and only if the reduced row echelon form of the augmented matrix has a pivot element in each column.

16. Every system of linear equations having no free variables is consistent.

17. Whenever a system of equations $\mathbf{Ax} = \mathbf{0}$ has two free variables, the system has infinitely many solutions.

18. The span of the set of rows in an $m \times n$ matrix $\mathbf{A}$ is the same as the set $\{\mathbf{Ax} : \mathbf{x} \in \mathbb{R}^n\}$.

19. The system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent if $\mathbf{x}$ is in the span of the set of columns of $\mathbf{A}$.

20. If the columns of a matrix $\mathbf{A}$ span $\mathbb{R}^k$, then for every vector $\mathbf{b}$ in $\mathbb{R}^k$ the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent.

21. If the vector $\mathbf{b}$ is *not* in the span of the set of columns of the matrix $\mathbf{A}$, then the system of equations $\mathbf{Ax} = \mathbf{b}$ is inconsistent.

22. If the vector $\mathbf{b}$ is in the span of the set of rows of a matrix $\mathbf{A}$, then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent.

23. If each column of an $m \times n$ matrix $\mathbf{A}$ has a pivot position, then the columns of $\mathbf{A}$ span $\mathbb{R}^m$.

24. Let $\mathbf{A}$ be an $m \times n$ matrix. If the equation $\mathbf{Ax} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$, then $\mathbf{A}$ has a pivot position in each row.

25. If a system of linear equations has free variables, then the system will be inconsistent.

26. These three vectors span $\mathbb{R}^3$:

$$\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

27. If a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $\mathbb{R}^m$, then every vector in $\mathbb{R}^m$ has a unique representation as a linear combination of $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$.

28. If $\mathbf{A}$ is an $m \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ has a solution.

29. Let $\mathbf{A}$ be an $m \times n$ matrix. If the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$, then $\mathbf{A}$ has a pivot position in each column.

30. A matrix can be row equivalent to two different matrices that are in row echelon form.

31. A system of linear equations is inconsistent if and only if the reduced row echelon form of the augmented matrix contains a row of the type $[a_1\ a_2\ \ldots a_n \mid 0]$, in which at least one of the terms a_i is not zero.

32. Consider two systems of linear equations, $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Cy} = \mathbf{d}$. If $\mathbf{A}$ is row equivalent to $\mathbf{C}$, then the two systems have the same solutions.

33. If all the entries in $\mathbf{A}$ and $\mathbf{b}$ are integers, and if the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent, then the solution vectors will have only integer components.

34. If a system of linear equations is inconsistent, then it will have infinitely many solutions.

35. If a system of linear equations is consistent, then it will have infinitely many solutions.

36. A system of linear equations is *consistent* if it has no solution.

37. If $\mathbf{A} \sim \mathbf{C}$ and $\mathbf{A} \sim \mathbf{D}$, then $\mathbf{C} \sim \mathbf{D}$. (Use $\sim$ for row equivalence of matrices.)

38. Two different matrices can be row equivalent to a third matrix that is in reduced row echelon form.

39. A matrix can be row equivalent to two different matrices that are in reduced row echelon form.

40. Let $[\mathbf{A} \mid \mathbf{b}]$ be the augmented matrix of a system of linear equations. If every row of this augmented matrix has a pivot position, then the system is consistent.

41. Let $[\mathbf{A} \mid \mathbf{b}]$ be the augmented matrix of a system of linear equations. If, after the row reduction to reduced row echelon form, every column of the resulting augmented matrix has a pivot, then the system is consistent.

42. A pivot position in a matrix is a location in the matrix where a leading 1 will appear when the matrix is put into reduced row echelon form.

43. In the matrix shown, the zero element is in a pivot position: $\begin{bmatrix} 0 & 3 & 6 & 4 \\ 1 & 3 & 7 & 4 \\ 2 & 3 & 5 & 3 \\ 1 & 4 & 5 & 9 \end{bmatrix}$

44. There exist examples of matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, such that $\mathbf{A}$ is row equivalent to $\mathbf{C}$ and $\mathbf{B}$ is row equivalent to $\mathbf{C}$, but $\mathbf{A}$ is not row equivalent to $\mathbf{B}$.

45. If the matrix $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right]$ is the augmented matrix of a consistent system of equations, then $h = 2$.

46. If $\mathbf{A}$ is a $p \times q$ matrix and if $q > p$, then every equation of the form $\mathbf{Ax} = \mathbf{b}$ (where $\mathbf{b} \in \mathbb{R}^p$) will have infinitely many solutions.

47. If the matrix $\mathbf{A}$ has more rows than columns, then for some vectors $\mathbf{b}$ the system $\mathbf{Ax} = \mathbf{b}$ will be inconsistent. (Assume that $\mathbf{A}$ is $m \times n$ and the $\mathbf{b}$ vectors will be in $\mathbb{R}^m$.)

48. The equation $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if the corresponding system of equations has at least one free variable.

49. Let the rows of the matrix $\mathbf{A}$ be denoted by $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ Then $\mathbf{Ax}$ is $x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n$.

50. Let $[\mathbf{A} \mid \mathbf{b}]$ be the augmented matrix of a system of linear equations. If every row of $\mathbf{A}$ has a pivot position, then the system is consistent.

51. Let **A** be an $m \times n$ matrix, and let p be the number of pivot positions in **A**. Only one of these conclusions is justified by the hypotheses: **a.** $p = n$, **b.** $p \geq n$, **c.** $p \geq m$, **d.** $2p \leq n + m$

52. If **A** is row equivalent to **B**, then for any vector **b**, the equations $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Bx} = \mathbf{b}$ have the same solutions.

53. The solution of a system of equations has been described by $x_1 = 3 + 2x_3$, $x_2 = 5 + x_3$, $x_4 = 2 - 4x_3$, where x_3 is a free variable. An alternative description is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \\ -4 \end{bmatrix}$$

54. The vector $\begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix}$ is a linear combination of the columns of the matrix $\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

55. The set of all vectors having the form $(a - 2b,\ b - a,\ 3b - 5a)$ is the span of the set $\{(1, -1, -5), (-2, 1, 3)\}$.

56. If an $m \times n$ matrix **A** has fewer than n pivots (or *pivot positions*), then for any vector **b** in $\mathbb{R}^m$, there will be infinitely many solutions to the equation $\mathbf{Ax} = \mathbf{b}$.

57. If an $m \times n$ matrix **A** has n pivots (or *pivot positions*), then for any vector **b** in $\mathbb{R}^m$, there will be finitely many solutions to the equation $\mathbf{Ax} = \mathbf{b}$.

58. The vector $(3, 1, 11)$ is in the span of the set of vectors $(3, 2, 5)$, $(1, 1, 1)$, $(0, 0, 1)$.

59. If the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent, then **b** is not in the span of the set of rows of **A**.

60. The matrix equation $\mathbf{Ax} = \mathbf{b}$, where **A** is $m \times n$, means that $\sum_{j=1}^{n} a_{ji}x_j = b_i$ for $1 \leq i \leq m$. (Here, a_{pq} are elements of the matrix **A** and b_k are the components of the vector **b**.

61. This matrix product is correctly carried out:

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \\ 31 \end{bmatrix}$$

62. The vector $(12, 1, -21)$ is in the span of this set of three vectors: $\{(11, 3, 7), (0, 12, 5), (0, 0, 66)\}$.

63. This equation is correct:

$$\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$$

64. If $h \neq 0$, this augmented matrix corresponds to an inconsistent system of equations $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & 4 & -5 & 14 \\ 2 & 2 & 11 & h \end{array}\right]$

65. A linear system of equations is inconsistent if and only if the reduced row echelon form of its augmented matrix has a row of the form $[0\ 0 \cdots 0 \mid 1]$.

66. When $h = 6$, this system is consistent:

$$\begin{cases} x_1 + 2x_2 = 3 \\ 3x_1 + hx_2 = 5 \end{cases}$$

67. The system of equations having this augmented matrix is consistent:

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & 6 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & -9 & 0 \end{array}\right]$$

68. Whenever a system of linear equations has free variables, the solution set contains many solutions.

69. The general solution of $x_1 + 3x_2 - 7x_3 = 0$ can be written

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

70. The span of the set of columns in an $n \times m$ matrix $\mathbf{A}$ is the same as the set $\{\mathbf{Ax} : \mathbf{x} \in \mathbb{R}^n\}$.

71. The matrix equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is $m \times n$, means that $\sum_{p=1}^{n} a_{qp}x_p = b_q$ for $1 \leq q \leq m$. (Here, a_{qp} are elements of the matrix $\mathbf{A}$ and b_q are the components of the vector $\mathbf{b}$.)

72. Let $\mathbf{A}$ be an $m \times n$ matrix, where $n > m$. If $\mathbf{Ax} = \mathbf{b}$ for some $\mathbf{x}$ and $\mathbf{b}$, then for some $\mathbf{y}$ different from $\mathbf{x}$, we have $\mathbf{Ay} = \mathbf{b}$.

73. If $\mathbf{A}$ is an $m \times n$ matrix and $m < n$, then the equation $\mathbf{Ax} = \mathbf{0}$ will have infinitely many solutions.

74. If the columns of a matrix $\mathbf{A}$ span $\mathbb{R}^k$, then for every vector $\mathbf{b}$ in $\mathbb{R}^k$ the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent.

75. Whenever a system of equations $\mathbf{Ax} = \mathbf{0}$ has a free variable, the system is consistent.

76. Whenever a system of equations $\mathbf{Ax} = \mathbf{0}$ has no free variables, the system has a unique solution.

77. Consider two systems of linear equations, $\mathbf{Ay} = \mathbf{b}$ and $\mathbf{Bx} = \mathbf{b}$, where $\mathbf{b}$ is the same in both equations. If $\mathbf{A}$ is row equivalent to $\mathbf{B}$, then the two systems have different solutions when $\mathbf{A} \neq \mathbf{B}$.

78. If the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is in the span of the set of rows of $\mathbf{A}$.

79. This matrix product is correctly carried out:

$$\begin{bmatrix} 1 & 5 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 37 \\ 13 \\ 32 \end{bmatrix}$$

80. The vector $(31, 43, -51)$ is in the span of this set of three vectors: $(11, 3, 7)$, $(0, 12, 5)$, $(0, 0, 66)$.

81. If $\mathbf{C} \sim \mathbf{A}$, $\mathbf{C} \sim \mathbf{D}$, and $\mathbf{E} \sim \mathbf{D}$, then $\mathbf{E} \sim \mathbf{A}$. (Use $\sim$ for row equivalence of matrices.)

82. The system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent if $\mathbf{x}$ is in the span of the set of rows of $\mathbf{A}$.

83. If the set of columns in a matrix $\mathbf{A}$ spans $\mathbb{R}^k$, then for every vector $\mathbf{b}$ in $\mathbb{R}^k$ the system of equations $\mathbf{Ax} = \mathbf{b}$ is inconsistent.

MULTIPLE-CHOICE EXERCISES 1.2

Always select the first correct answer.

1. When $\mathbf{A} = \begin{bmatrix} 4 & 3 & 4 \\ 2 & -1 & 0 \\ 1 & 6 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ the third component of $\mathbf{Ax}$ is
   a. 22 **b.** 16 **c.** 13 **d.** 8
   e. None of these.

2. For what value of α is the vector $(2, 11, -3)$ in the span of the set $\{(2, 5, -3), (4, 8, \alpha)\}$?
   a. 4 **b.** 1 **c.** -6 **d.** 2
   e. None of these.

3. What is the span of the set $\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (2, 1)$? (These are vectors in $\mathbb{R}^2$.)
   a. A line through the points $\mathbf{u}$ and $\mathbf{v}$.
   b. The set of all scalar multiples of $\mathbf{u}$.
   c. The set of all vectors of the form $\mathbf{u} + t\mathbf{v}$, where t is a real parameter.
   d. $\mathbb{R}^2$
   e. None of these.

4. Let $\mathbf{u} = (3, 7, 0)$, $\mathbf{v} = (-2, 3, 0)$, and $\mathbf{w} = (5, 5, 5)$. Which statement is true?
   a. $\mathbf{u} \in \text{Span}\{\mathbf{v}, \mathbf{w}\}$ **b.** $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$
   c. $\mathbf{v} \in \text{Span}\{\mathbf{u}, \mathbf{w}\}$ **d.** None of these.

5. Let $\mathbf{u} = (3, 2, 1)$, $\mathbf{v} = (1, 2, 3)$, $\mathbf{w} = (1, 1, 1)$. Which statement is false?
   a. $\mathbf{u} \in \text{Span}\{\mathbf{v}, \mathbf{w}\}$ **b.** $\mathbf{v} \in \text{Span}\{\mathbf{u}, \mathbf{w}\}$
   c. $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$ **d.** $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$
   e. None of these.

6. Let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (0, -1, 2)$, and $\mathbf{w} = (2, 7, 0)$. Which assertion is true?
   a. $\mathbf{u} \in \text{Span}\{\mathbf{v}, \mathbf{w}\}$ and $\mathbf{w} \notin \text{Span}\{\mathbf{u}, \mathbf{v}\}$
   b. $\mathbf{u} \in \text{Span}\{\mathbf{v}, \mathbf{w}\}$ and $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$
   c. $\mathbf{v} \in \text{Span}\{\mathbf{u}, \mathbf{w}\}$ and $\mathbf{u} \notin \text{Span}\{\mathbf{v}, \mathbf{w}\}$
   d. $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$ and $\mathbf{v} \notin \text{Span}\{\mathbf{u}, \mathbf{w}\}$
   e. None of these.

7. Which of these assertions is *not* logically equivalent to all the others? (The matrix $\mathbf{A}$ is $m \times n$.)
   a. The row vectors in $\mathbf{A}$ span $\mathbb{R}^n$.
   b. The column vectors in $\mathbf{A}$ span $\mathbb{R}^m$.
   c. For each $\mathbf{b} \in \mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ has a solution.
   d. $\mathbf{A}$ has a pivot position in each row.
   e. None of these.

8. Let $\mathbf{A}$ be an $m \times n$ matrix. In order to have a set of three or four logically equivalent statements, which one of these conditions must be deleted?
   a. The columns of $\mathbf{A}$ span $\mathbb{R}^m$.
   b. The rows of $\mathbf{A}$ span $\mathbb{R}^n$.
   c. For every $\mathbf{b}$ in $\mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ is consistent.
   d. Every row of $\mathbf{A}$ has a pivot position.
   e. None of these.

9. Consider the system of equations whose augmented matrix is $\left[\begin{array}{cc|c} 14 & 8 & a \\ 21 & 12 & b \end{array}\right]$
   Which assertion is correct?
   a. It has a solution for all choices of a and b.
   b. It is inconsistent for all choices of a and b.
   c. It has a solution for some choices of a and b.
   d. It has a solution only in the case $a = 0$.
   e. None of these.

10. For what value of c is the vector $(c, 20, -7)$ in the linear span of $(3, 7, 1)$ and $(4, -2, 3)$?
   a. 0 **b.** 5 **c.** -6 **d.** 7
   e. None of these.

11. Let $\mathbf{u} = (3, 0, 2)$, $\mathbf{v} = (1, 1, 1)$, and $\mathbf{w} = (3, -3, 1)$. Which vector is in Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?
a. $(1, 2, 3)$ **b.** $(4, 1, 2)$ **c.** $(5, -1, 3)$
d. $(11, 2, 9)$ **e.** None of these.

12. Let $\mathbf{u} = (3, 0, 2)$, $\mathbf{v} = (0, 1, 1)$, and $\mathbf{w} = (-3, 1, 0)$. Which vector is in Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?
a. $(1, 2, 3)$ **b.** $(4, 1, 2)$ **c.** $(5, -1, 5)$
d. $(11, 2, 9)$ **e.** None of these.

13. For what value of λ is the vector $(\lambda, 3, -5)$ in the linear span of $(1, 3, -1)$ and $(-5, -8, 2)$?
a. 3 **b.** -6 **c.** 0 **d.** -5
e. None of these.

14. For what value of h is the vector $(3, -5, h)$ in the linear span of $(1, 3, -1)$ and $(-5, -8, 2)$?
a. 3 **b.** -6 **c.** 0 **d.** -5
e. None of these.

15. Let $[\mathbf{A}\,|\,\mathbf{b}] = \left[\begin{array}{cccc|c} 2 & 0 & 2 & 8 & 11 \\ 7 & 1 & 2 & 8 & 35 \\ 2 & 0 & 1 & 4 & 9 \\ 1 & 0 & 2 & 8 & 7 \end{array}\right]$

Which system is *not* equivalent to this system?

a. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ **b.** $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 8 & 3 \end{array}\right]$

c. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 3 & 1 & 0 & 0 & 17 \\ 2 & 0 & 1 & 4 & 9 \\ 1 & 0 & 2 & 8 & 7 \end{array}\right]$ **d.** $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 7 & 1 & 2 & 8 & 35 \\ 2 & 0 & 1 & 4 & 9 \\ 1 & 0 & 2 & 8 & 7 \end{array}\right]$

e. None of these.

16. When $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

then $\mathbf{Ax}$ is: **a.** $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ **b.** $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

c. $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$

e. None of these.

17. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 4 & 5 & 1 \end{bmatrix}$

Which of these matrices is row equivalent to $\mathbf{A}$?

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. None of these.

18. Let $\mathbf{A} = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

Which of the following four matrices is not row equivalent to $\mathbf{A}$?

a. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 4 & 0 \end{bmatrix}$

e. None of these.

19. Which of these is *not* a permissible action on the rows of an augmented matrix, because it might change the solution set of a system of equations? (*Caution*: This is tricky!) (In each case assume $i \neq j$.)
a. $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$
b. $\mathbf{r}_i \leftarrow 2.3\mathbf{r}_i$
c. $\mathbf{r}_i \leftarrow 7\mathbf{r}_i + \mathbf{r}_j$
d. $\mathbf{r}_j \leftarrow 3.7\mathbf{r}_i + \mathbf{r}_j$
e. None of these.

20. Which system of equations has many solutions? In each case we show the augmented matrix of the system of equations.

a. $\left[\begin{array}{ccc|c} 2 & 5 & 9 & 3 \\ 0 & 3 & 7 & 5 \\ 0 & 0 & 4 & 1 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 2 & 5 & 9 & 3 \\ 4 & 3 & 7 & 5 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 2 & 5 & 3 & 1 \\ 6 & 15 & 9 & 5 \end{array}\right]$ **d.** $\left[\begin{array}{cccc|c} 0 & 2 & 5 & 3 & 1 \\ 0 & 6 & 15 & 9 & 5 \end{array}\right]$

e. None of these.

21. Let $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (1, 1, 0)$. Which of the following vectors is not in Span$\{\mathbf{u}, \mathbf{v}\}$?

a. $(1, -1, 2)$ **b.** $(4, 3, 1)$ **c.** $(1, 1, 1)$
d. $(1, 2, -1)$ **e.** None of these.

22. Let $\mathbf{A}$ be an $m \times n$ matrix. The set $\{\mathbf{A}\mathbf{x} : x \in \mathbb{R}^n\}$ is the same as

a. The set of row vectors in **A**.
b. The set of column vectors in **A**.
c. The span of the set of rows in **A**.
d. The span of the set of columns in **A**.
e. None of these.

23. Consider $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \end{array}\right]$

What is the general solution of the system with this *augmented* matrix?

a. $x_1 = -1, x_2 = 1, x_3 = 0, x_4 = -4, x_5 = 0$
b. $x_1 = -1 - x_5, x_2 = 1 + 3x_5, x_3 = \text{free}, x_4 = -4 - 5x_5, x_5 = \text{free}$
c. $x_1 = -3 - 7x_5 + 2x_2, x_2 = \text{free}, x_3 = \text{free}, x_4 = -4 - 5x_5, x_5 = \text{free}$
d. $x_1 + x_5 = -1, x_2 - 3x_5 = 1, x_4 + 5x_5 = -4, x_3 = \text{free}$
e. None of these.

24. What is $\mathbf{A}\mathbf{x}$ when $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$?

a. $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ **b.** $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

c. $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$

e. None of these.

25. Let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (0, -1, 2)$, and $\mathbf{w} = (2, 7, 0)$. Which assertion is false?

a. $\mathbf{u} \in$ Span$\{\mathbf{v}\}$ and $\mathbf{w} \notin$ Span$\{\mathbf{u}, \mathbf{v}\}$
b. $\mathbf{w} \in$ Span$\{\mathbf{u}, \mathbf{v}\}$ **c.** $\mathbf{w} \notin$ Span$\{\mathbf{u}, \mathbf{v}\}$
d. $\mathbf{v} \notin$ Span$\{\mathbf{u}, \mathbf{w}\}$ **e.** None of these.

26. Consider $\left[\begin{array}{cccccc|c} 0 & 4 & 3 & 4 & 6 & 11 & 5 \\ 0 & 0 & 0 & 7 & 2 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 \end{array}\right]$

Suppose a system of linear equations has the *augmented* matrix shown. What are all the free variables?

a. x_2, x_4, x_6 **b.** x_3, x_5 **c.** x_1, x_3, x_5
d. x_1, x_3, x_5, x_7 **e.** None of these.

27. Let $\mathbf{A} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$

What are all the pivot positions in this matrix?

a. a_{11}, a_{22}, a_{33}
b. a_{11}, a_{23}, a_{35}
c. a_{11}, a_{23}
d. a_{11}, a_{23}, a_{35}
e. None of these.

28. Which matrix is in row echelon form?

$\mathbf{A} = \begin{bmatrix} 7 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 5 & 2 & 4 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 7 & 3 & 2 & 3 \\ 0 & 5 & 4 & 6 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 0 & 5 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\mathbf{D} = \begin{bmatrix} 3 & 5 & 7 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$

a. D **b.** A and C **c.** B **d.** C
e. None of these.

29. Which matrix is in reduced row echelon form?

$$A = \begin{bmatrix} 3 & 5 & 7 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 5 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & 3 & 2 & 3 \\ 0 & 5 & 4 & 6 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 7 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 5 & 2 & 4 \end{bmatrix}$$

a. D **b.** A **c.** D and B
d. C **e.** None of these.

30. Which matrix is the augmented matrix of an inconsistent system?

$$A = \begin{bmatrix} 7 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 5 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 5 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & 3 & 2 & 3 \\ 0 & 5 & 4 & 6 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 5 & 7 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

a. B **b.** C **c.** A and C
d. B and D **e.** None of these.

31. Which statement is incorrect?

a. Vector $(3, 4, 7)$ is in the span of $(2, 1, 0)$ and $(1, 3, 5)$.
b. Vector $(3, 1, 2)$ is in the span of $(1, 4, 2)$, $(3, -1, 2)$ and $(5, 7, 5)$.
c. Vector $(5, 5, 5)$ is in the span of $(2, 1, 0)$ and $(1, 3, 5)$.
d. Vector $(1, -7, -15)$ is in the span of $(1, 3, 5)$ and $(2, 1, 0)$.
e. None of these.

32. Which expression is *not* defined?

a. $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

c. $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$

e. None of these.

33. Describe all solutions of this linear system:

$$\begin{cases} 3x_1 + 6x_2 + x_3 + 3x_4 = -9 \\ 2x_1 + 4x_2 + x_3 + 3x_4 = 7 \end{cases}$$

a. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -16 \\ 0 \\ 39 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -16 \\ 0 \\ 39 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} -16 \\ 0 \\ 39 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} = 0$

d. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -16 \\ 0 \\ 39 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

e. None of these.

34. Let $\mathbf{A} = \begin{bmatrix} 3 & 4 & 4 \\ -1 & 0 & 2 \\ 6 & 2 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

The third component of **Ax** is

a. 22 **b.** 16 **c.** 13 **d.** 8
e. None of these.

35. Let **A** be an $m \times n$ matrix, and let p be the number of pivot positions in **A**. Only one of these conclusions is justified by the hypotheses. Which one?

a. $p = n$ **b.** $p \geq n$ **c.** $p \geq m$
d. $2p \leq n + m$ **e.** None of these.

36. The system $x + y + z = 6, x + 2y + 3z = 14, y + 2z = 8$
a. Has an infinite number of solutions.
b. Is inconsistent.
c. Has the unique solution $(-2, 8, 0)$.
d. Has a finite number of solutions.
e. None of these.

37. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

the system $\mathbf{Ax} = \mathbf{b}$ has
a. No solution.
b. Only the solution $[-1, 1, 1, 1]^T$.
c. Only the solution $[-2, 2, 7, 3]^T$.
d. Two solutions.
e. The parametric solution $[-1, 0, 1, 1]^T + c[0, 1, 0, 0]^T$ with c free.
f. None of these.

38. If the augmented matrix of a system contains the row $[0\,0\,0\,0\,0 \mid 1]$, we can conclude that the system
a. Has a unique solution.
b. Has many solutions.
c. Has at least 5 free variables.
d. Is inconsistent.
e. None of these.

39. If the augmented matrix of a system of equations has $[0 \quad 0 \quad 0 \quad 0 \quad 0 \mid -91]$ as its first row, what conclusion can be drawn about the system?
a. It is inconsistent.
b. It has a unique solution.
c. It has many solutions.
d. It has at least 2 free variables.
e. None of these.

40. Consider $\left[\begin{array}{cc|c} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 4 \end{array}\right]$

What is the solution of the system having this augmented matrix?
a. System has no solution. **b.** $(-5, 6)$
c. $(-7, 1, 4)$ **d.** $(5, -6)$
e. None of these.

41. Let $\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 2 & -8 & 5 \end{array}\right]$ be an augmented matrix. This system has how many solutions?
a. None
b. At least two
c. At most one
d. Exactly one
e. None of these.

COMPUTER EXERCISES 1.2

1. (Roundoff Error.) Solve the following system of equations by the Gaussian elimination method, using Matlab or some other similar system:

$$\begin{cases} 3.277x_1 + 5.113x_2 = 2.237 \\ 1.482x_1 + 2.321x_2 = 4.209 \end{cases}$$

Then do the same when all real numbers in the input data are rounded to three significant figures. Draw conclusions.

2. In Matlab, one can ask for calculated quantities to be expressed as quotients of integers by the command `format rat`. Using this command, find the solution of this system expressed as quotients of integers.

$$\left[\begin{array}{rrr|r} 5 & 1 & -2 & 2 \\ 3 & -1 & 4 & 1 \\ 2 & 7 & 6 & 3 \end{array}\right]$$

3. Use mathematical software to find the general solutions to General Exercises 6, 9, 18–19, 23, 31–32, and 38–40 in this section.

4. Find the reduced row echelon forms for the two matrices in General Exercise 15.

5. Use mathematical software to solve General Exercise 21 in its original form without using a change of variable.

1.3 KERNELS, RANK, HOMOGENEOUS EQUATIONS

> *Each problem that I solved became a rule which served afterwards to solve other problems.*
>
> –RENÉ DESCARTES (1596–1650)
>
> *There are no solved problems. There are only problems that are more or less solved.*
>
> JULES HENRÍ POINCARÉ (1854–1912)

At this point in the book, it is assumed that the reader knows all about the reduced row echelon form of a matrix (in particular how to compute it) and what *pivot* elements are.

Kernel or Null Space of a Matrix

A system of linear equations of the form $\mathbf{Ax} = \mathbf{0}$ is said to be **homogeneous**. (In the study of differential equations the same concept arises.) This is a special case of the general system that we usually write as $\mathbf{Ax} = \mathbf{b}$. A first observation about a homogeneous system is that we can take $\mathbf{x} = \mathbf{0}$ as a solution. This is called the **trivial solution**. The issue, then, is whether there are any other solutions. If so, they are called **nontrivial solutions**. Our goal is to find, for any specific matrix $\mathbf{A}$, a complete description of the set of all solutions of its homogeneous equation. That set of vectors is

$$\text{Ker}(\mathbf{A}) = \{\mathbf{x} : \mathbf{Ax} = \mathbf{0}\} = \text{Null}(\mathbf{A})$$

The abbreviation **Ker** is for the word **kernel**. The notation Null($\mathbf{A}$) is also used, where **Null** is an abbreviation for **null space**. Thus, the kernel or null space of a matrix is the set of all vectors that are mapped into $\mathbf{0}$ by the mapping $\mathbf{x} \mapsto \mathbf{Ax}$. If $\mathbf{A}$ is an $m \times n$ matrix, the kernel of $\mathbf{A}$ is a subset of $\mathbb{R}^n$. In symbols, $\text{Ker}(\mathbf{A}) \subseteq \mathbb{R}^n$ or $\text{Null}(\mathbf{A}) \subseteq \mathbb{R}^n$. This set is never empty, is it?

EXAMPLE 1 Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \end{bmatrix}$. Is the vector $\mathbf{v} = [7, 6, -5]^T$ in the kernel of this matrix?

SOLUTION Yes, one has only to verify that $\mathbf{Av} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ -5 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 5\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \\ 14 \end{bmatrix} + \begin{bmatrix} 18 \\ 6 \end{bmatrix} + \begin{bmatrix} -25 \\ -20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{x}$ is a nonzero vector such that $\mathbf{Ax} = \mathbf{0}$, then we have a nontrivial equation of the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Here the $\mathbf{a}_i$ terms are the columns of the matrix $\mathbf{A}$. If $\mathbf{A}$ is $m \times n$, then each column is a vector in $\mathbb{R}^m$. The preceding equation exhibits a linear relation among the columns of $\mathbf{A}$. In this case, it is a nontrivial equation, because $\mathbf{x} \neq \mathbf{0}$. (In other words, the vector $\mathbf{x}$ has at least one nonzero component.) When this occurs, we say that the set of columns of $\mathbf{A}$ is **linearly dependent**. Otherwise, we say that the set of columns of $\mathbf{A}$ is **linearly independent**. This important terminology is explained in detail on p. 89.

THEOREM 1

If $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* Ker($\mathbf{A}$), *and if* α *is a scalar, then* $\mathbf{x} + \mathbf{y}$ *and* $\alpha\mathbf{x}$ *are also in* Ker($\mathbf{A}$).

PROOF Here is the proof for addition: Assume the hypotheses. Then

$$\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$ ■

THEOREM 2

If $\mathbf{Ax} = \mathbf{b}$ *and* $\mathbf{y} \in$ Ker($\mathbf{A}$), *then* $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{b}$.

THEOREM 3

If $\mathbf{Ax} = \mathbf{Ay}$, *then* $\mathbf{x} - \mathbf{y} \in$ Ker($\mathbf{A}$).

THEOREM 4

If $\mathbf{u}$ *is a vector such that* $\mathbf{Au} = \mathbf{b}$, *then every solution of the equation* $\mathbf{Ax} = \mathbf{b}$ *is of the form* $\mathbf{x} = \mathbf{u} + \mathbf{z}$, *for some vector* $\mathbf{z}$ *in* $\text{Ker}(\mathbf{A})$.

The proofs of Theorems 2, 3, and 4 are straightforward.

THEOREM 5

If two matrices are row equivalent to each other, then their kernels are the same.

PROOF If $\mathbf{A}$ and $\mathbf{B}$ are two matrices that are row equivalent to each other, then the solutions of $\mathbf{Ax} = \mathbf{0}$ and the solutions of $\mathbf{Bx} = \mathbf{0}$ are the same. This is because row operations performed on a system of equations do not alter the solutions. Hence, the kernels of $\mathbf{A}$ and $\mathbf{B}$ are the same. ■

Homogeneous Equations

Theorems 1 to 5 hint at the utility of knowing all solutions to a homogeneous equation $\mathbf{Ax} = \mathbf{0}$. A typical example follows.

EXAMPLE 2 We seek a description of the set of all solutions to the equation $\mathbf{Ax} = \mathbf{0}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 3 & 10 & -7 & 1 \\ -5 & -5 & 3 & 7 \end{bmatrix}$$

SOLUTION The row reduction of the augmented matrix leads to

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & 0 & 0 \\ 3 & 10 & -7 & 1 & 0 \\ -5 & -5 & 3 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Notice that x_4 is a free variable. Furthermore, the last column of 0's in the augmented matrix could be omitted in such a calculation because it remains a zero column throughout the row-reduction process. The corresponding system of equations can be written as

$$\begin{cases} x_1 = 2x_4 \\ x_2 = 0 \\ x_3 = x_4 \end{cases}$$

We write this in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where t is a free parameter. In words: The kernel of $\mathbf{A}$ consists precisely of all scalar multiples of the vector $[2, 0, 1, 1]^T$. This is a line in four-space ($\mathbb{R}^4$) passing through $\mathbf{0}$. ■

Uniqueness of the Reduced Row Echelon Form

The logical underpinning of row reduction, as a technique applied to matrices in general, depends on the following theorem.

THEOREM 6

Every matrix has one and only one reduced row echelon form.

PROOF Let $\mathbf{A}$ be $m \times n$. The proof uses induction on n. For $n = 1$, the theorem is obvious. Suppose, now, that the theorem has been established for $m \times (n-1)$ matrices. Let $\mathbf{A}$ be an $m \times n$ matrix. Denote by $\mathbf{A}_{n-1}$ the matrix obtained from $\mathbf{A}$ by removing its nth column. Any sequence of row operations that brings $\mathbf{A}$ to reduced row echelon form also puts $\mathbf{A}_{n-1}$ in reduced row echelon form. By the induction hypothesis, $\mathbf{A}_{n-1}$ has one and only one reduced row echelon form. If $\mathbf{B}$ and $\mathbf{C}$ are reduced row echelon forms of $\mathbf{A}$, they can differ only in the nth column. Assume $\mathbf{B} \neq \mathbf{C}$. Select i so that row i in $\mathbf{B}$ differs from row i in $\mathbf{C}$. Thus, we have $b_{in} \neq c_{in}$. Because $\mathbf{A} \sim \mathbf{B} \sim \mathbf{C}$, the homogeneous systems $\mathbf{Ax} = \mathbf{0}$, $\mathbf{Bx} = \mathbf{0}$, and $\mathbf{Cx} = \mathbf{0}$ have the same solutions. Let $\mathbf{Bu} = \mathbf{0}$. Then $\mathbf{Cu} = \mathbf{0}$ and $(\mathbf{B} - \mathbf{C})\mathbf{u} = \mathbf{0}$. The first $n-1$ columns of $\mathbf{B}-\mathbf{C}$ are all zeros. Hence, we have $[(\mathbf{B}-\mathbf{C})\mathbf{u}]_i = (b_{in} - c_{in})u_n = 0$. Since $b_{in} \neq c_{in}$, we must have $u_n = 0$. Thus, any solution of $\mathbf{Bx} = \mathbf{0}$ or $\mathbf{Cx} = \mathbf{0}$ must have $x_n = 0$. It follows that the nth column of $\mathbf{B}$ and $\mathbf{C}$ must have pivots, for otherwise those columns would be associated with free variables, and we could choose x_n to be nonzero. Because the first $n-1$ columns of $\mathbf{B}$ and $\mathbf{C}$ are identical, the row in which this pivotal 1 appears must be the same for $\mathbf{B}$ and $\mathbf{C}$; namely, it is the row which is the first zero row of the reduced row echelon form of $\mathbf{A}_{n-1}$. Because the remaining entries in the nth columns of $\mathbf{B}$ and $\mathbf{C}$ must be zero, we have $\mathbf{B} = \mathbf{C}$, which is a contradiction. This proof follows the one given by Yuster [1984]. ■

Rank of a Matrix

An important concept in linear algebra is the **rank** of a matrix. It is directly related to the number of pivot positions in the matrix, and, thus, not surprisingly, it rests upon the precedng theorem. ■

DEFINITION

*The **rank** of a matrix is the number of nonzero rows in its reduced row echelon form. We use the notation* **Rank**(**A**) *for this number.*

This definition depends logically on our acknowledging that a given matrix has a unique reduced row echelon form. Remember that in mathematics the word *unique* does not mean *unusual* or *noteworthy*; it means one of a kind. (In non-scientific and non-mathematical contexts, casual use of the word *unique* has degraded it to the point where it has almost *no* meaning.)

The computation of the rank of a matrix can be done by obtaining its reduced row echelon form and counting the number of nonzero rows in the result. In fact, it is not necessary to get the *reduced* row echelon form of the given matrix; its rank will be evident from any row echelon form of the matrix. This is true because a row echelon form of a matrix reveals how many nonzero rows there will be in the reduced row echelon form.

EXAMPLE 3 What is the rank of this matrix?

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 2 & 6 & -2 & 14 & -3 & -19 \\ 4 & 12 & 2 & 16 & 1 & 7 \end{bmatrix}$$

SOLUTION In the row-reduction process, we show a row echelon form and the reduced row echelon form:

$$\mathbf{A} \sim \begin{bmatrix} \boxed{1} & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & \boxed{8} & -16 & 7 & 55 \\ 0 & 0 & 0 & 0 & \boxed{-2} & -62 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 0 & 5 & 0 & -1 \\ 0 & 0 & \boxed{1} & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

That the rank is 3 can be concluded from either of these two row echelon forms. Here the pivot positions are the boxed entries. In particular, one need not carry out the reduction to the *reduced* row echelon form. ■

Recall that the locations of the pivots are called **pivot positions** for the given matrix.

EXAMPLE 4 What are the pivot positions in the matrix of Example 3?

SOLUTION The pivot positions are $(1,1)$, $(2,3)$, and $(3,5)$. Notice that the definition of pivot positions refers to the reduced row echelon form of the matrix, but the location of those pivot positions is already clear from any row echelon form. ■

In the reduced row echelon form, each pivot must be to the right of and below the position of the previous pivot. In other words, each new pivot must account for a nonzero row and a nonzero column because of the required pattern.

COROLLARY 1

The rank of a matrix is the number of pivots in its reduced row echelon form, which is the same as the number of pivot positions in the matrix.

EXAMPLE 5 What are the pivot positions in the matrix $\begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 4 & 6 & 3 \\ 3 & 3 & 7 & 5 \end{bmatrix}$?

SOLUTION By carrying out a few row operations, we obtain

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 4 & 6 & 3 \\ 3 & 3 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & -9 & -11 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 4 & 6 & 3 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & \boxed{5} & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 0 & 7/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 3/5 \\ 0 & \boxed{1} & 0 & 1/5 \\ 0 & 0 & \boxed{1} & 1/5 \end{bmatrix}$$

We see that the pivot positions are $(1,1)$, $(2,2)$, and $(3,3)$, which are shown as boxed entries. The matrix has rank 3. Note especially that the pivot positions cannot be predicted solely from the numbers in the original matrix. For example, the entry 0 in the original matrix turns out to occupy a pivot position. ■

THEOREM 7

The rank of an $m \times n$ matrix cannot be greater than n or m. In symbols, we have $\text{Rank}(\mathbf{A}) \leq \min(m, n)$.

PROOF Think about the reduced row echelon form of an $m \times n$ matrix. Each nonzero row must have a pivot element, and these must occur in the staircase pattern. In counting the pivots, we note that each new pivot must account for a nonzero row and a nonzero column. The count of all pivots cannot therefore exceed n or m. Because the rank is the number of pivots, the rank can be at most n and at most m. ■

The next theorem elaborates on this theme. It concerns matrices of size $m \times n$ that have rank less than n. The theorem states that if an $m \times n$ matrix has any one of the properties labeled **a**, **b**, . . . , **f**, then it must have all those properties.

THEOREM 8

These properties of an $m \times n$ matrix $\mathbf{A}$ *are equivalent to each other.*

- **a.** *The rank of* $\mathbf{A}$ *is less than* n.
- **b.** *The reduced row echelon form of* $\mathbf{A}$ *has fewer than* n *nonzero rows.*
- **c.** *The matrix* $\mathbf{A}$ *has fewer than* n *pivot positions.*
- **d.** *At least one column in* $\mathbf{A}$ *has no pivot position.*
- **e.** *There is at least one free variable in the system of equations* $\mathbf{Ax} = \mathbf{0}$.
- **f.** *The system* $\mathbf{Ax} = \mathbf{0}$ *has some nontrivial solutions.*

A useful corollary of Theorem 8 involves only the dimensions of a matrix:

COROLLARY 2

A homogeneous system of linear equations in which there are more variables than equations must have some nontrivial solutions.

PROOF Let the system have the form $\mathbf{Ax} = \mathbf{0}$, where the matrix $\mathbf{A}$ is $m \times n$. The homogeneous system of equations has m equations and n unknowns. Therefore, by hypothesis, $n > m$. It follows from Theorem 7 that

$$\text{Rank}(\mathbf{A}) \leq \min(n, m) = m < n$$

Apply Theorem 8 to conclude that the homogeneous system has nontrivial solutions. ■

General Solution of a System

The next few examples illustrate techniques for finding the general solution of a system of equations.

EXAMPLE 6 What do the Theorems 7 and 8 and Corollary 2 tell us about this system?

$$\begin{cases} x_1 + 3x_2 + 9x_3 = 0 \\ 2x_1 + 7x_2 + 3x_3 = 0 \end{cases}$$

SOLUTION In this system, there are more variables than equations. Therefore, by Corollary 2, the system has nontrivial solutions. Using Theorem 7, we find that Rank($\mathbf{A}$) $\leq \min\{2, 3\} = 2$. The row-reduction process applied to the augmented matrix shows that

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 0 \\ 2 & 7 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 54 & 0 \\ 0 & \boxed{1} & -15 & 0 \end{array}\right]$$

There are two pivots, and the rank of the matrix is 2. There is one column without a pivot, and this indicates the presence of a free variable. (Here, it is x_3.) With x_3 assigned any value, we have $x_1 = -54x_3$ and $x_2 = 15x_3$. This is the *general* solution of the homogeneous system. The recommended form of the *general solution* is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -54 \\ 15 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -54 \\ 15 \\ 1 \end{bmatrix}$$

where t is a free parameter. Thus, all solutions to the homogeneous equation are scalar multiples of the vector $[-54, 15, 1]^T$. All these points lie on a line through the origin in three-space ($\mathbb{R}^3$). ■

EXAMPLE 7 Find all the solutions of this system of linear equations:

$$\begin{cases} 2x_1 - 4x_2 = 3 \\ 4x_1 - x_2 = 2 \\ x_1 - x_2 = 1 \end{cases}$$

SOLUTION The row-reduction algorithm leads to this conclusion:

$$\left[\begin{array}{cc|c} 2 & -4 & 3 \\ 4 & -1 & 2 \\ 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

Of course, we are working with the augmented matrix. The final row of the row-reduced matrix states that $0x_1 + 0x_2 = 1$. This precludes the existence of a solution. ■

The system of equations in Example 7 is *inconsistent*. Notice that there is a pivot position in the last column of this augmented matrix. A moment's thought will convince us that this is always the sign of an inconsistent system. Indeed, if there is a pivot in the last column, there must be zeros elsewhere in that row. Hence, the equation corresponding to that row is of the form $0x_1 + 0x_2 + \cdots + 0x_n = 1$, which is not possible!

Matrix–Matrix Product

In Section 1.2, the matrix–vector product

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

was defined. Now we give meaning to the matrix–matrix product **AB** whenever **A** is an $m \times n$ matrix and **B** is an $n \times k$ matrix. Notice the requirement that the number of columns in **A** matches the number of rows in **B**.

DEFINITION

If **A** *is an* $m \times n$ *matrix and* **B** *is an* $n \times k$ *matrix, then the* **matrix–product AB** *is defined to be the* $m \times k$ *matrix whose columns are* $\mathbf{Ab}_1, \mathbf{Ab}_2, \ldots, \mathbf{Ab}_k$*. Here the vectors* $\mathbf{b}_i$ *are the columns of matrix* **B**.

One can write the defining equation in this form:

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_k \end{bmatrix}$$

where each column of **AB** can be computed as a linear combination of the columns of **A**.

$$\mathbf{Ab}_i = b_{i1}\mathbf{a}_1 + b_{i2}\mathbf{a}_2 + \cdots + b_{in}\mathbf{a}_n$$

Special cases of the definition arise when $m = 1$ or $k = 1$. In these cases, we get formulas for $\mathbf{y}^T\mathbf{B}$ and $\mathbf{Ax}$ where $\mathbf{y}^T$ is a $1 \times n$ row vector and $\mathbf{x}$ is an $n \times 1$ column vector. When $m = k = 1$, we get the dot product $\mathbf{y}^T\mathbf{x}$, which is a 1×1 scalar.

EXAMPLE 8 As an illustration, we carry out the important process of matrix–matrix multiplication of a 2×2 matrix $\mathbf{A}$ and a 2×3 matrix $\mathbf{B}$ in full detail:

$$\begin{aligned}\mathbf{AB} &= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & -1 \\ 2 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right]\end{aligned}$$

SOLUTION

$$\begin{aligned}&= \left[\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right] \\ &= \left[1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}, 4\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}, -1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right] \\ &= \left[\begin{bmatrix} 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 16 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 7 & 16 & 3 \\ 3 & 6 & 2 \end{bmatrix}\end{aligned}$$

Here we have inserted some commas for clarity. ■

EXAMPLE 9 What is the numerical value of the following product?

$$\mathbf{AB} = \begin{bmatrix} 1 & 3 & 2 & -5 \\ 2 & 2 & -3 & 4 \\ 5 & 1 & 1 & 6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -5 & 0 \\ 4 & 1 \end{bmatrix}$$

SOLUTION Such a product is *not* defined because the matrices $\mathbf{A}$ and $\mathbf{B}$ are incompatible! The product $\mathbf{AB}$ of two matrices will exist if and only if the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{B}$. Here we have a 3×4 matrix $\mathbf{A}$ and a 2×2 matrix $\mathbf{B}$. Remember, if $\mathbf{A}$ is $m \times n$, then $\mathbf{B}$ must be $n \times k$, for some value of k. The values of m and k are unrestricted. ■

In Section 3.1, the topic of matrix–matrix multiplication is explored in more detail.

EXAMPLE 10 How can we efficiently solve two systems, $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{c}$, when the coefficient matrix $\mathbf{A}$ is the same in the two systems?

SOLUTION This example shows how to solve several systems of equations that differ only in their righthand sides. To solve $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{c}$, we can set up two augmented matrices $[\mathbf{A} \mid \mathbf{b}]$ and $[\mathbf{A} \mid \mathbf{c}]$. Then we carry out the row reduction of both augmented matrices. However, because $\mathbf{A}$ is the same in both, it is more efficient to create this augmented matrix $[\mathbf{A} \mid \mathbf{b} \mid \mathbf{c}]$ and carry out the row reduction on it. ■

EXAMPLE 11 For a concrete case of this technique, solve the systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{c}$, when

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 7 & -11 \\ 2 & -4 & 1 & 1 \\ 1 & 2 & -5 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 9 \\ -5 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 9 \\ -9 \end{bmatrix}$$

SOLUTION The augmented matrix and its reduced row echelon form are

$$[\,\mathbf{A} \mid \mathbf{b} \mid \mathbf{c}\,] = \left[\begin{array}{cccc|c|c} 1 & 3 & 7 & -11 & 6 & 1 \\ 2 & -4 & 1 & -1 & 9 & 9 \\ 1 & 2 & -5 & 2 & -5 & -9 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c|c} \boxed{1} & 0 & 0 & -1 & 2 & 0 \\ 0 & \boxed{1} & 0 & -1 & -1 & -2 \\ 0 & 0 & \boxed{1} & -1 & 1 & 1 \end{array}\right]$$

We find the general solution for $\mathbf{Ax} = \mathbf{b}$ to be

$$\mathbf{x} = \mathbf{u} + s\mathbf{z} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and the general solution for $\mathbf{Ay} = \mathbf{c}$ to be

$$\mathbf{y} = \mathbf{v} + t\mathbf{z} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Here s and t are free parameters. The work can be verified by these three equations:

$$\begin{bmatrix} 1 & 3 & 7 & -11 \\ 2 & -4 & 1 & 1 \\ 1 & 2 & -5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 7 & -11 \\ 2 & -4 & 1 & 1 \\ 1 & 2 & -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 7 & -11 \\ 2 & -4 & 1 & 1 \\ 1 & 2 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

■

EXAMPLE 12 Solve the matrix equation $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ -3 & 2 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 & 13 & 15 & -5 \\ 9 & 15 & 14 & 2 \\ 1 & 10 & 15 & 1 \end{bmatrix}$$

Also, verify the solution in an independent manner.

SOLUTION This again illustrates the important technique of solving systems of equations with a single coefficient matrix but multiple righthand sides. In this example, we face a problem of solving four linear systems $\mathbf{Ax}^{(i)} = \mathbf{b}^{(i)}$, for $i = 1, 2, 3, 4$, each with the same coefficient matrix $\mathbf{A}$:

$$\mathbf{AX} = \mathbf{A}[\mathbf{x}^{(1)},\ \mathbf{x}^{(2)},\ \mathbf{x}^{(3)},\ \mathbf{x}^{(4)}] = [\mathbf{b}^{(1)},\ \mathbf{b}^{(2)},\ \mathbf{b}^{(3)},\ \mathbf{b}^{(4)}] = \mathbf{B}$$

Because the matrix $\mathbf{A}$ is 3×3 and the matrix $\mathbf{B}$ is 3×4, the matrix $\mathbf{X}$ must be 3×4. The augmented matrix for the problem is

$$[\mathbf{A} \mid \mathbf{B}] = \left[\begin{array}{rrr|rrrr} 1 & 3 & 1 & 6 & 13 & 15 & -5 \\ 2 & 1 & 4 & 9 & 15 & 14 & 2 \\ -3 & 2 & 5 & 1 & 10 & 15 & 1 \end{array}\right]$$

The row-reduction process leads to these matrices:

$$[\mathbf{A} \mid \mathbf{B}] \sim \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 3 & 4 & -2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 \end{array}\right] = [\mathbf{I} \mid \mathbf{X}]$$

where

$$\mathbf{X} = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 3 & 4 & -2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

An independent verification is possible as follows:

$$\mathbf{AX} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 3 & 4 & -2 \\ 1 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 13 & 15 & -5 \\ 9 & 15 & 14 & 2 \\ 1 & 10 & 15 & 1 \end{bmatrix} = \mathbf{B}$$

■

In Section 3.2, the techniques used in this problem will be called upon again, for computing right and left inverses of matrices.

Indexed Sets of Vectors: Linear Dependence and Independence

In describing sets of vectors, we usually think of the vectors as having **indices** attached to them. For example, we might write

$$\{\mathbf{u}_1 = (7, 3), \quad \mathbf{u}_2 = (6, -4), \quad \mathbf{u}_3 = (4, 11)\}$$

Here, the first vector has the index number 1 associated with it, and the other two vectors have index numbers 2 and 3 associated with them. Trouble arises, however, if we have repetitions in the definition of a set, such as this:

$$\{(7, 3), \quad (6, -4), \quad (7, 3)\}$$

Is this a set of two vectors or three vectors? As an ordinary set it has only two elements because the third one mentioned is the same as the first. The *set* is the same as

$$\{(7, 3), \quad (6, -4)\}$$

The fact that one vector is mentioned twice does not mean that we have three vectors, because one is equal to another. However, in this example, the difficulty is avoided, since these three indexed vectors are deemed to be different (because their indices differ) in the *indexed set:*

$$\{\mathbf{u}_1 = (7, 3), \quad \mathbf{u}_2 = (6, -4), \quad \mathbf{u}_3 = (7, 3)\}$$

As a consequence, the columns of the matrix

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 7 \\ 3 & -4 & 3 \end{bmatrix}$$

are regarded as being different, since they have (invisible) indices 1, 2, 3. In a few moments, we will be saying that the columns of this matrix form a linearly dependent set, and the validity of this assertion depends on our thinking of the columns forming an indexed set.

DEFINITION

An indexed set of vectors, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, *is* **linearly dependent** *if there exists a nontrivial equation of the form* $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$. *In the contrary case, the indexed set is* **linearly independent**. *Nontrivial in this context means* $\sum_{i=1}^{n} |c_i| > 0$.

EXAMPLE 13 Let $\mathbf{u}_1 = (3, 7, 4)$, $\mathbf{u}_2 = (-4, 2, 2)$, $\mathbf{u}_3 = (0, 17, 11)$. Is this set of three vectors linearly dependent?

SOLUTION We are asking whether there is a nontrivial equation of the form

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 17 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This equation is the same as

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}\mathbf{c} = \begin{bmatrix} 3 & -4 & 0 \\ 7 & 2 & 17 \\ 4 & 2 & 11 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A row reduction on the coefficient matrix leads to a row echelon form:

$$\begin{bmatrix} 3 & -4 & 0 \\ 7 & 2 & 17 \\ 4 & 2 & 11 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 0 \\ 1 & 10 & 17 \\ 1 & 6 & 11 \end{bmatrix} \sim \begin{bmatrix} 0 & -34 & -51 \\ 1 & 10 & 17 \\ 0 & -4 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 10 & 17 \\ 0 & 34 & 51 \\ 0 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 17 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 11 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving the homogeneous system with the resulting matrix, we have $c_1 = -2c_3$ and $2c_2 = -3c_3$. This reveals that there exist nontrivial solutions to the homogeneous problem. If we set the free variable c_3 equal to 2, for example, we get a nontrivial solution $c_1 = -4$, $c_2 = -3$, and $c_3 = 2$. Thus, the original set of three vectors is linearly dependent: $-4\mathbf{u}_1 - 3\mathbf{u}_2 + 2\mathbf{u}_3 = -4(3, 7, 4) - 3(-4, 2, 2) + 2(0, 17, 11) = (0, 0, 0) = \mathbf{0}$. ■

EXAMPLE 14 By direct use of the definition of linear dependence, determine whether these (indexed) sets are linearly independent or linearly dependent. In the case of linear dependence, give the coefficients that establish that fact.

a. $\{\mathbf{u}_1 = (1, 3, 6),\ \mathbf{u}_2 = (2, 7, 5),\ \mathbf{u}_3 = (0, 0, 0)\}$

b. $\{\mathbf{z}_1 = (7, 6, 3),\ \mathbf{z}_2 = (5, 2, 1),\ \mathbf{z}_3 = (7, 6, 3)\}$

c. $\{\mathbf{x}_1 = (7, 6),\ \mathbf{x}_2 = (5, 4),\ \mathbf{x}_3 = (14, 12)\}$

d. $\{\mathbf{v}_1 = (1, 3),\ \mathbf{v}_2 = (2, 7),\ \mathbf{v}_3 = (4, 13)\}$

e. $\{\mathbf{w}_1 = (7, 2, 3),\ \mathbf{w}_2 = (-1, 1, 0),\ \mathbf{w}_3 = (1, 3, -1)\}$

SOLUTION The first set, **a**, is linearly dependent because it contains the zero vector and $0\mathbf{u}_1 + 0\mathbf{u}_2 + 1\mathbf{u}_3 = \mathbf{0}$. The second set, **b**, illustrates a repeated entry in an indexed set. The set is linearly dependent because $\mathbf{z}_1 - \mathbf{z}_3 = 0$. The third set, **c**, is linearly dependent by inspection because $2\mathbf{x}_1 + 0\mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$. The fourth set, **d**, is also linearly dependent because $2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. If you did not immediately notice this relationship, the reduced row echelon form could be used to reveal it. However, the linear dependence (without the coefficients) can be predicted most efficiently by Corollary 2 (p. 82). It asserts that an indexed set of $n + 1$ vectors in $\mathbb{R}^n$ is necessarily linearly dependent. The fifth set, **e**, is linearly independent because the reduction process yields

$$\begin{bmatrix} \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 7 \\ 1 & 3 & 2 \\ 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -7 \\ 0 & 4 & 9 \\ 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

There is a pivot position in each column of the original matrix and the only solution of the homogeneous system is the trivial one. ■

Here is an algorithm for determining whether a set of vectors in $\mathbb{R}^m$ is linearly independent.

ALGORITHM

Given an (indexed) set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$, form a matrix $\mathbf{A}$ using these n vectors as columns. If the equation $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution, then the given set of vectors is linearly dependent. If the equation has only the $\mathbf{0}$ solution, then the set is linearly independent.

Another example to illustrate the testing for linear independence follows. It also shows that the property of linear dependence can be sensitive to small changes in the data. The set of three vectors $\{(1, 2, 1), (3, 1, 1), (5, 5, 3)\}$ is linearly dependent because

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -5 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we make a small change in one vector. The new set of vectors is $\{(1, 2, 1), (3, 1, 1)(5, 5, 3.01)\}$ and it is linearly independent because

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 5 \\ 1 & 1 & 3.01 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -5 \\ 0 & -2 & -1.99 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

THEOREM 9

If an indexed set of two or more vectors is linearly dependent, then some vector in the set is a linear combination of the others.

PROOF Consider the equation $\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0}$. If the equation has a nontrivial solution, then some coefficient is nonzero. For simplicity, assume $c_1 \neq 0$. Then $\mathbf{v}_1 = -(1/c_1) \sum_{i=2}^{k} c_i \mathbf{v}_i$. ■

THEOREM 10

If an indexed set of two or more vectors in $\mathbb{R}^m$ is linearly dependent, then some vector in the list is a nontrivial linear combination of vectors preceding it in the list.

PROOF If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent, we have $\sum_{i=1}^{p} c_i\mathbf{v}_i = \mathbf{0}$ in a nontrivial equation. Let c_q be the last nonzero coefficient. Then $\sum_{i=1}^{q} c_i\mathbf{v}_i = \mathbf{0}$ and $c_q \neq 0$. Solving for $\mathbf{v}_q$, we have $\mathbf{v}_q = (-1/c_q)\sum_{i=1}^{q-1} c_i\mathbf{v}_i$. ■

EXAMPLE 15 Consider the matrix $\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix}$. Can the parameter h be chosen so that the three columns form a linearly independent set?

SOLUTION In this matrix, let the columns be $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. Obviously, we have $2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, and the set of columns is linearly dependent. No value of h makes $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent. ■

THEOREM 11

The rows of a matrix form a linearly dependent set if and only if there is a zero row in any row echelon form of that matrix.

PROOF Let $\mathbf{A}$ be the matrix under consideration, and let its rows be denoted by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. In the following list of assertions, each implies the one following:

a. The set $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$ is linearly dependent.
b. A nontrivial equation $\sum_{i=1}^{m} c_i\mathbf{r}_i = \mathbf{0}$ is true.
c. For some index k, we have $c_k \neq 0$ and $\mathbf{r}_k + \sum_{i\neq k}(c_i/c_k)\mathbf{r}_i = \mathbf{0}$.
d. Row $\mathbf{r}_k$ becomes $\mathbf{0}$ if we add to it suitable multiples of the other rows.
e. Any row echelon form of $\mathbf{A}$ has a zero row.

For the converse, assume that $\mathbf{A}$ is row equivalent to a matrix $\mathbf{B}$ that has a zero row, say row $\mathbf{r}_k$. That zero row arises from adding multiples of rows in $\mathbf{A}$ onto row $\mathbf{r}_k$ and from moving the rows so that the zero rows are at the bottom. This process is the same as the one described in the first half of the proof of this theorem, and we can conclude that the rows of $\mathbf{A}$ form a linearly dependent set. ■

Using the Row-Reduction Process

By applying the row-reduction process to a matrix, we can easily decide whether the set of rows is linearly dependent or linearly independent. If a row echelon form has a zero row, then the rows form a linearly dependent set, and vice versa. If we want the coefficients that make the equation

$\sum_{i=1}^{n} c_i\mathbf{r}_i = \mathbf{0}$ true, we can get them from the row-reduction process. Here is an example of this technique.

EXAMPLE 16 Determine whether the rows of this matrix form a linearly dependent set, and if so, find the coefficients in a nontrivial combination that is $\mathbf{0}$.

$$\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 2 \\ -1 & 1 & 3 & -3 \\ -2 & -2 & 1 & -5 \end{bmatrix}$$

SOLUTION We immediately consider $\mathbf{A}^T$, whose columns are the rows of $\mathbf{A}$. Its reduced row echelon form is shown here:

$$\mathbf{A}^T = \begin{bmatrix} 1 & -1 & -2 \\ 3 & 1 & -2 \\ 2 & 3 & 1 \\ 2 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have obtained the linear combination $c_1\mathbf{r}_1+c_2\mathbf{r}_2+c_3\mathbf{r}_3 = \mathbf{0}$, where $c_1 = c_3$ and $c_2 = -c_3$. Consequently, one nontrivial solution is $c_1 = 1, c_2 = -1, c_3 = 1$. Check: $(1, 3, 2, 2) - (-1, 1, 3, -3) + (-2, -2, 1, -5) = \mathbf{0}$. ■

The proofs of the following two theorems are left as exercises.

THEOREM 12

The column vectors of a matrix form a linearly dependent set if and only if there is a column having no pivot.

THEOREM 13

The column vectors of a matrix form a linearly independent set if and only if there is a pivot position in each column of the matrix.

EXAMPLE 17 Use Theorems 11 and 12 to determine whether this set of vectors is linearly dependent or linearly independent:

$$\{(1, 3, 7), (2, 5, -4), (-5, -11, 37)\}$$

SOLUTION First, form a matrix with the three indicated vectors as its rows. Then carry out a row-reduction process to see whether the reduced row echelon form has a zero row. (Any row echelon form of the matrix will serve to answer this question.) We have

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 5 & -4 \\ -5 & -11 & 37 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & -18 \\ 0 & 4 & 72 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 18 \\ 0 & 1 & 18 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 18 \\ 0 & 0 & 0 \end{bmatrix}$$

We conclude from Theorem 11 that the given set of three vectors is linearly dependent.

Next, we carry out the row-reduction process of a matrix with the three indicated vectors as its columns:

$$\begin{bmatrix} 1 & 2 & -5 \\ 3 & 5 & -11 \\ 7 & -4 & 37 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & 4 \\ 0 & -18 & 72 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & 4 \\ 0 & -1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there is a column without a pivot, these column vectors form a linearly dependent set, by Theorem 12. In particular, we find $-3(1, 3, 7) + 4(2, 5, -4) = (-5, -11, 37)$. We can find these coefficients for this linear combination only from the second system—*not* the first one! Why? Notice that

$$-3(1, 3, 7) + 4(2, 5, -4) = (-5, -11, 37)$$

and

$$-3\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \\ 37 \end{bmatrix}$$

In determining whether a set of vectors is linearly dependent, the vectors can be taken to be either the rows or the columns of a matrix. ■

Determining Linear Dependence or Independence

When solving small problems by hand, one can determine whether a set of n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^m$ is linearly independent or linearly dependent as follows.

1. By inspection, determine whether the set contains $\mathbf{0}$. If so, one concludes immediately that the set is linearly dependent.
2. Does the set contain two vectors, of which one is a multiple of the other? If so, the set is linearly dependent.
3. If it is evident that some vector in the set is a linear combination of other vectors in the set, then the set is linearly dependent.
4. Use Theorems 8, 12, and 13 or Corollary 2. Typically, this will involve putting the vectors as columns in an $m \times n$ matrix. If $n > m$, then the set of columns is linearly dependent, by Corollary 2. In this case, no calculation is needed, only counting!
5. Look at the general case if none of the preceding is true. Put the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as columns into a matrix $\mathbf{A}$. Carry out the row-reduction process on $\mathbf{A}$ to obtain a row echelon form. Either the system $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution or it does not. In the first case, the set of columns is linearly dependent. Otherwise, the set of columns in $\mathbf{A}$ is linearly independent.

In other words, first try to find special cases that immediately solve the problem before launching into the general case, which may be long and tedious.

Application: Chemistry

A typical *unbalanced* equation describing a chemical reaction is

$$\mathbf{B_2S_3} + \mathbf{H_3N} \rightarrow \mathbf{B_3N_2} + \mathbf{S_3H_4}$$

(This equation may not describe a possible reaction: it is used only to illustrate the principles.) The capital letters refer to various chemical elements: $\mathbf{B}$ = Boron, $\mathbf{N}$ = Nitrogen, $\mathbf{S}$ = Sulfur, and $\mathbf{H}$ = Hydrogen. The equation can be put into words as follows: If the compound $\mathbf{B_2S_3}$ reacts with the compound $\mathbf{H_3N}$, the result will be two compounds, $\mathbf{B_3N_2}$ and $\mathbf{S_3H_4}$. What is missing in this assertion is information about the relative quantities of each compound involved in the reaction. To balance the equation, numerical factors must be associated with each compound, and when that has been done we will know the relative amounts of each compound participating in the chemical reaction. Here we are balancing the number of atoms in each element. Once these numbers have been determined, the relative masses of the elements in the reaction can be computed. Thus, after the balancing process, the number of atoms of each element should be the same on the two sides of the equation.

To proceed, we associate a factor with each compound. These factors are unknown at the beginning of our analysis, and are denoted by x, y, z, w. Now we write

$$x\mathbf{B}_2\mathbf{S}_3 + y\mathbf{H}_3\mathbf{N} = z\mathbf{B}_3\mathbf{N}_2 + w\mathbf{S}_3\mathbf{H}_4$$

Counting the number of atoms for each element, **B**, **S**, **H**, **N**, leads to four equations as follows:

$$\mathbf{B}: \ 2x = 3z \qquad \mathbf{S}: \ 3x = 3w \qquad \mathbf{H}: \ 3y = 4w \qquad \mathbf{N}: \ y = 2z$$

If all terms are placed on the lefthand side, we obtain four homogeneous equations with four unknowns. The coefficient matrix and its partially reduced form are

$$\begin{bmatrix} 2 & 0 & -3 & 0 \\ 3 & 0 & 0 & -3 \\ 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A convenient solution for the homogeneous system is $x = 3$, $y = 4$, $z = 2$, and $w = 3$. A balanced equation is

$$3\mathbf{B}_2\mathbf{S}_3 + 4\mathbf{H}_3\mathbf{N} = 2\mathbf{B}_3\mathbf{N}_2 + 3\mathbf{S}_3\mathbf{H}_4$$

This work reveals the ratios of the compounds in the reaction. For example, the number of molecules of $\mathbf{H}_3\mathbf{N}$ should be four-thirds the number of molecules of $\mathbf{B}_2\mathbf{S}_3$, if the chemical reaction is to use all the material provided. After the reaction has taken place, the number of molecules of $\mathbf{S}_3\mathbf{H}_4$ should be three-halves the number of molecules of $\mathbf{B}_3\mathbf{N}_2$, and so on.

SUMMARY 1.3

- Homogeneous systems: $\mathbf{Ax} = \mathbf{0}$
- Trivial solution $\mathbf{x} = \mathbf{0}$;
  nontrivial solution $\mathbf{x} \neq \mathbf{0}$
- Kernel or Null Space:
  $\text{Ker}(\mathbf{A}) = \text{Null}(\mathbf{A}) = \{\mathbf{x} : \mathbf{Ax} = \mathbf{0}\}$
- $\text{Ker}(\mathbf{A}) = \text{Null}(\mathbf{A}) \subseteq \mathbb{R}^n$ if $\mathbf{A}$ is $m \times n$
- The columns of $\mathbf{A}$ form a linearly independent set if and only if the equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution
- The columns of $\mathbf{A}$ form a linearly dependent set if and only if the equation $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution

- A linear combination of columns of $\mathbf{A}$ can be written as $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$ or as $\mathbf{Ax}$. The vectors $\mathbf{a}_i$ are the columns of $\mathbf{A}$.
- Theorems on kernel (or null space):
  - If $\mathbf{x}$ and $\mathbf{y}$ are in Null$(\mathbf{A})$, then so are $\mathbf{x}+\mathbf{y}$ and $\alpha\mathbf{x}$
  - If $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{y} \in \text{Ker}(\mathbf{A})$, then $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{b}$
  - If $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$, then $\mathbf{x} - \mathbf{y}$ is in Ker$(\mathbf{A})$
  - If $\mathbf{A} \sim \mathbf{B}$, then Ker$(\mathbf{A})$ = Ker$(\mathbf{B})$
- A given matrix has one and only one reduced row echelon form
- The rank of a matrix is the number of rows that have pivot positions
- If $\mathbf{A}$ is an $m \times n$ matrix, then Rank$(\mathbf{A}) \le \min\{m, n\}$
- One can solve the equations $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{c}$ by using the augmented matrix $[\mathbf{A} \mid \mathbf{b}\ \mathbf{c}]$
- If $\mathbf{A}$ is an $m \times n$ matrix, then the number of nonzero rows in its reduced row echelon form is at most n
- If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}$ has at most n pivot positions
- If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times q$, then $\mathbf{AB} = [\mathbf{Ab}_1\ \mathbf{Ab}_2\ \cdots, \mathbf{Ab}_q]$

KEY CONCEPTS 1.3

Homogeneous systems, trivial solutions and nontrivial solutions, kernel of a matrix, null space of a matrix, row-equivalent matrices, rank, pivot position, unicity of the reduced row echelon form, pivot positions in a matrix, upper bounds on the rank of a matrix, some equivalent properties of matrices, matrix–matrix multiplication, an inconsistent system and its reduced row echelon form, indexed set, linear independence, linear dependence, chemical application

GENERAL EXERCISES 1.3

1. Solve this system of equations by carrying out the reduction of the augmented matrix to reduced row echelon form:

$$\begin{cases} 5x_1 - 2x_2 = 9 \\ 3x_1 - x_2 = 8 \\ 11x_1 - 3x_2 = 33 \end{cases}$$

2. (Continuation.) Compute the rank of the coefficient matrix.

3. Find the general solution of this system of equations and express it in the manner recommended in the text:

$$\begin{cases} x_1 + 3x_2 + 9x_3 = 6 \\ 2x_1 + 7x_2 + 3x_3 = -5 \\ x_1 + 4x_2 - 6x_3 = -11 \end{cases}$$

4. For the matrix shown here, compute its rank and find a set of vectors whose span is its kernel:

$$\begin{bmatrix} 1 & 4 & -5 & 10 \\ 3 & 1 & 7 & -3 \\ 2 & 2 & 2 & 2 \\ 1 & 3 & -3 & 7 \end{bmatrix}$$

5. Let $\mathbf{A} = \begin{bmatrix} 7 & 2 & 0 & 49 & -37 \\ 3 & 1 & 0 & 22 & 16 \\ 4 & 2 & 2 & 31 & 20 \end{bmatrix}$

Find a pair of vectors whose span is Ker($\mathbf{A}$).

6. Find a matrix whose kernel is spanned by the two vectors $\mathbf{u} = (1, 3, 2)$ and $\mathbf{v} = (-2, 0, 4)$.

7. Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 7 \\ 2 & -4 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 11 & 6 & 2 & -12 & 27 \\ -1 & 9 & -20 & -5 & -19 \\ -2 & -5 & 21 & 22 & 9 \end{bmatrix}$

Solve the system of equations $\mathbf{AX} = \mathbf{B}$. Then do the same for the following augmented matrix: $\left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

8. This problem is solved readily with the technique explained in Example 10 (p. 86). We ask: What are the vectors $\mathbf{u}$ and $\mathbf{v}$ if $(-12, 10, 20) = 3\mathbf{u} + 5\mathbf{v}$ and $(17, -8, 21) = 5\mathbf{u} - 4\mathbf{v}$?

9. Explain why the system $\mathbf{Ax} = \mathbf{0}$ is always consistent.

10. Without any calculations, provide nontrivial solutions to the equation $\mathbf{Ax} = \mathbf{0}$, when $\mathbf{A}$ is in turn each of the following matrices:

a. $\begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 7 & 7 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -14 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & 9 \\ 2 & 9 & 13 \end{bmatrix}$

11. (Continuation.) For the three matrices, find nontrivial vectors in their kernels. No calculations are necessary.

12. Explain why the following two matrices are *not* row equivalent to each other by showing that the corresponding systems of homogeneous equations have different solutions (assume $a \neq b$):

$\mathbf{A} = \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. Establish that if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$.

14. Find two vectors whose span is the kernel of the matrix $\begin{bmatrix} 7 & 3 & 5 & 37 \\ 2 & 1 & 1 & 11 \end{bmatrix}$

15. Solve these two systems of equations in an efficient manner: $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$

16. If $\mathbf{A} = \begin{bmatrix} x & 1 & 0 \\ -9 & y & 7 \\ -1 & 4 & z \end{bmatrix}$ and the kernel of $\mathbf{A}$ contains the vector $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, what are x, y, and z?

17. Determine whether this is true:

$\begin{bmatrix} 1 & 3 & 0 & 5 & 0 & 4 \\ 2 & 6 & 1 & 8 & 0 & 5 \\ 2 & 6 & 2 & 6 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

18. Determine whether this set of vectors is linearly dependent:

$\{(3, 2, 7), (4, 1, -3), (6, -1, -23)\}$

19. Explain why a system of equations $\mathbf{Ax} = \mathbf{b}$ has either no solution, exactly one solution, or infinitely many solutions. Explain how these three outcomes are easily distinguished after the row reduction of the augmented matrix has been carried out.

20. Justify, without appealing to the reduced row echelon form, the assertion that if a system of equations $\mathbf{Ax} = \mathbf{b}$ has two solutions then it has infinitely many solutions.

21. (Continuation.) Consider a system of equations $\mathbf{Ax} = \mathbf{b}$, and assume that it has two solutions, say $\mathbf{u}$ and $\mathbf{v}$. Explain why, for all real values of t, $t\mathbf{v} + (1 - t)\mathbf{u}$ is also a solution. Establish then that the solution set of the system contains a line.

22. Establish the validity of Theorems 1–4 (pp. 77–78).

23. Explain why two matrices that are row equivalent to each other must have the same rank.

24. Explain why a set of n vectors in $\mathbb{R}^m$ is linearly independent if and only if the matrix having these vectors as its columns has rank n.

25. Establish directly that if $\mathbf{Ax} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$, then the rank of $\mathbf{A}$ is less than n. (Here $\mathbf{A}$ is $m \times n$.)

26. Let the matrix $\mathbf{A}$ be in reduced row echelon form. Explain why each nonzero row contains a pivot element. Is the same assertion true for the columns of $\mathbf{A}$?

27. Explain why the rank of $\mathbf{A}$ is the number of pivot positions in $\mathbf{A}$.

28. Establish that the rank of $\mathbf{A}$ is the number of columns that contain pivot positions.

29. Consider a consistent system of equations $\mathbf{Ax} = \mathbf{b}$, in which $\mathbf{A}$ is $m \times n$ and $m < n$. Explain why the system must have many solutions.

30. Compute the ranks of these matrices:

a. $\begin{bmatrix} 5 & 2 & 0 & 18 \\ 2 & 1 & 0 & 8 \\ 3 & 3 & -1 & 15 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 4 & 3 & 7 & 5 & 4 \\ 0 & 2 & 2 & -1 & 6 \\ 0 & 0 & 5 & 2 & 3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

c. $\begin{bmatrix} e & 3 & 7 & 5 & 4 \\ 2 & 2 & -1 & 6 & 0 \\ 5 & 2 & 3 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}$

31. Let $\mathbf{A}$ be an $m \times n$ matrix whose kernel is $\mathbf{0}$; that is, the only solution of the equation $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. What is the rank of $\mathbf{A}$?

32. **a.** The linear system $x - y - z = 0$, $x + y - z = 0$ has infinitely many solutions. These are the points on the line of intersection of the two given planes in $\mathbb{R}^3$. Find the equation for this line.
b. Find a simple description of the set of points satisfying these three equations: $y+2z = 0$, $2x-y+8z = 0$, $x-y+3z = 0$.

33. Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 3 \end{bmatrix}$

What are the pivot positions in this matrix?

34. If $\mathbf{A}$ is an $m \times n$ matrix and $m \geq n$, what is the maximum number of pivot positions in $\mathbf{A}$? Explain. What is the maximum number of pivotal rows that $\mathbf{A}$ can have?

35. Let $\mathbf{A}$ be an $m \times n$ matrix, where $m < n$. What is the maximum number of pivot positions in $\mathbf{A}$? What is the least number of nonpivotal columns in $\mathbf{A}$? What is the least number of free variables in solving the equation $\mathbf{Ax} = \mathbf{0}$?

36. (Continuation.) Adopt the hypotheses on **A** as in the preceding question. Explain why the equation $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution. Explain why the columns of **A** form a linearly dependent set of vectors.

37. Let $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

If each of two matrices **A** and **B** is row equivalent to **C**, does it follow that $\mathbf{A} = \mathbf{B}$?

38. If two matrices **A** and **B** are *not* row equivalent to each other, can they have the same reduced row echelon form? Explain.

39. Let $\mathbf{u} = (1, 3, 2)$, $\mathbf{v} = (2, -1, 4)$, and $\mathbf{w} = (-3, 26, -6)$. Determine whether the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

40. (Challenging.) Consider this infinite sequence of matrices:

$$\mathbf{A}_1 = [1] \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \text{ and so on}$$

(We have shown only the first four of them.) Find the ranks of all of them.

41. If the rank of an augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ is greater than the rank of **A**, what conclusion can be drawn? Is there an implication in both directions?

42. (Challenging.) What are the ranks of the matrices in this infinite sequence?

$$\mathbf{A}_1 = [1] \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 9 & 10 & 11 & 12 \\ 16 & 15 & 14 & 13 \end{bmatrix} \text{ and so on}$$

43. Describe all the 3×4 matrices of rank 1.

44. Explain why, for any system of linear equations, the rank of the augmented matrix is at least as great as the rank of the coefficient matrix.

45. (Challenging.) Let $n \geq 3$, and create an $n \times n$ matrix **A** by defining $\mathbf{A}_{ij} = \alpha i + \beta j + \gamma$, where α, β, and γ are three arbitrary positive numbers. What is the rank of **A**?

46. Define a family of functions f_n by the equation $f_n(x) = 1$ when $x \geq n$ and $f_n(x) = 0$ if $x < n$. Is this family linearly independent? (Here $n = 0, 1, 2$ and so on.)

47. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent pair of vectors, is the same true for $\{\mathbf{v}_1, \mathbf{v}_2 + \lambda\mathbf{v}_1\}$ when λ is an arbitrary constant?

48. If $\{\mathbf{v}_1, \mathbf{v}_2 + \lambda\mathbf{v}_1\}$ is linearly independent for some nonzero scalar λ, does it follow that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent?

49. If $\{\mathbf{v}_1, \mathbf{v}_2 + \lambda\mathbf{v}_1\}$ is linearly dependent for some nonzero scalar λ, does it follow that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent?

50. Balance this chemical reaction:
$\mathbf{AgNO_3 + NaCl \rightarrow AgCl + NaNO_3}$

51. Balance this chemical reaction:
$\mathbf{H_2 + NO_2 \rightarrow NH_3 + H_2O}$

52. Balance this hypothetical chemical equation:
$\mathbf{NHCO_3 + HC \rightarrow NC + H_2O + CO_2}$

53. Consider $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 1 \\ 3 & 6 & h \end{bmatrix}$

For what value of h does this matrix have a nontrivial kernel?

54. Find the general solution of this system and display it in the recommended form:

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 3x_1 + 4x_2 - 3x_3 = 0 \end{cases}$$

55. Consider the vectors $\mathbf{u}_1 = (1, 3, 2)$, $\mathbf{u}_2 = (-2, 1, 4)$, and $\mathbf{u}_3 = (8, 3, -8)$. Taken alone, each of these vectors is linearly independent (which means in this case that each vector is nonzero). Hence, the English language allows us to say that they are linearly independent. Reconcile this conclusion with the easily verified fact that $2\mathbf{u}_1 - 3\mathbf{u}_2 - \mathbf{u}_3 = \mathbf{0}$. What is the remedy for this apparent inconsistency?

56. (Challenging.) Consider three vectors in $\mathbb{R}^2$: $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, and $\mathbf{w} = (w_1, w_2)$. Show that the following six conditions are incompatible: $u_1v_1 + u_2v_2 = 0$, $v_1w_1 + v_2w_2 = 0$, $w_1u_1 + w_2v_2 = 0$, $u_1^2 + u_2^2 > 0$, $v_1^2 + v_2^2 > 0$, $w_1^2 + w_2^2 > 0$.

57. Consider $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 1 & 4 & 6 \end{bmatrix}$

Find the rank of this matrix. What can be said of the equation $\mathbf{Ax} = \mathbf{0}$?

58. What conditions must be placed on $\mathbf{A}$ in order that the system $\mathbf{Ax} = \mathbf{0}$ be consistent?

59. Consider the system

$$\begin{cases} x_1 + 3x_2 + x_3 = 6 \\ 2x_1 + 6x_2 + 3x_3 = 16 \\ 3x_1 + 9x_2 + 4x_3 = 22 \end{cases}$$

Show the original augmented matrix. Obtain its reduced row echelon form. Give the rank of the coefficient matrix. Describe the solution of the system. Identify the independent (free) variables. Find all solutions of $\mathbf{Ax} = \mathbf{0}$ when $\mathbf{A}$ is the coefficient matrix.

60. Let $\mathbf{v}_1 = (1, 5, -2, 4)$, $\mathbf{v}_2 = (-1, 2, 2, -4)$, $\mathbf{v}_3 = (2, 12, -1, 12)$, $\mathbf{v}_4 = (0, 1, 1, 2)$, and $\mathbf{v}_5 = (3, -1, 4, -2)$. Is this set of vectors linearly independent? Explain fully.

61. Explain that if the equation $\mathbf{AX} = \mathbf{B}$ has more than one solution, then the equation $\mathbf{AX} = \mathbf{0}$ has a nontrivial solution. Establish that this equation has infinitely many solutions. Here $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{X}$ is an $n \times q$ matrix, and $\mathbf{B}$ is an $m \times q$ matrix.

62. If possible, express $(7, -1, 0)$ as a linear combination of $(1, 3, 2)$ and $(4, 1, 1)$. Explain how you solve this.

63. Suppose that the equation $\mathbf{Ax} = \mathbf{b}$ has more than one solution. Explain why the equation $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions.

64. (Challenging.) For each n there is a matrix $\mathbf{A}_n$ following this pattern:

$$\mathbf{A}_1 = [1] \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix} \text{ and so on.}$$

What are the ranks of these matrices?

65. Consider the system

$$\begin{cases} 4x_1 + 12x_2 + 6x_3 = 32 \\ 3x_1 + 9x_2 + 4x_3 = 22 \\ 4x_1 + 12x_2 + 4x_3 = 24 \end{cases}$$

Show the accompanying augmented matrix. Obtain the reduced row echelon form. Give the rank of the coefficient matrix. Describe the solutions of the system. Identify the independent (free) variables, if there are any.

66. (Continuation.) Find all solutions of $\mathbf{Ax} = \mathbf{0}$ when $\mathbf{A}$ is the coefficient matrix of the preceding problem.

67. Give a simple example where a system has a free variable and yet *no* solutions.

68. If $\text{Rank}(\mathbf{A}) = k$, what is $\text{Rank}([\mathbf{A} \,|\, \mathbf{b}])$?

69. Can our theory of linear equations be built up with just this one row operation: $\mathbf{r}_i \leftarrow \alpha\mathbf{r}_i + \beta\mathbf{r}_j$ for $i \neq j$ and for nonzero scalars α and β?

70. Establish that every matrix of rank r is a sum of r matrices of rank 1.

71. Establish the validity of
a. Theorem 8 (p. 82).
b. Theorem 12 (p. 93).
c. Theorem 13 (p. 93).

72. Consider $\{p_0, p_1, p_2, p_3\}$. Determine whether this set of polynomials is linearly independent or linearly dependent. The definitions are $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = 4 - t$, $p_3(t) = t^3$.

73. Let $f(t) = \sin t$ and $g(t) = \cos t$. Determine whether the pair $\{f, g\}$ is linearly dependent or linearly independent.

74. Let $f(t) = 1$, $g(t) = \cos 2t$, and $h(t) = \sin^2 t$. Determine whether the set $\{f, g, h\}$ is linearly dependent or independent. The domain of the functions is taken to be $\mathbb{R}$.

75. Test each of these three sets of functions $\{u_1, u_2, u_3\}$ for linear dependence or linear independence:
a. $u_1(t) = 1,\ u_2(t) = \sin t,\ u_3(t) = \cos t$
b. $u_1(t) = 1,\ u_2(t) = \sin^2 t,\ u_3(t) = \cos^2 t$
c. $u_1(t) = \cos 2t,\ u_2(t) = \sin^2 t,$
$u_3(t) = \cos^2 t$

76. Verify that there exists no 2×2, noninvertible, nonsymmetric matrix $\mathbf{A}$ such that $\text{Ker}(\mathbf{A}) = \text{Ker}(\mathbf{A}^T)$.

77. If the rank of an $m \times n$ matrix is less than m, can we conclude that the rows form a linearly dependent set? (The term *rank* is defined on p. 80.)

78. Establish that if a set of at least two vectors is linearly dependent, then one element of the set is a linear combination of the others.

79. Argue that a pair of vectors is linearly dependent if and only if one of the vectors is a multiple of the other.

80. Explain why a pair of vectors is linearly independent if the two vectors are not colinear with the $\mathbf{0}$-vector.

81. Explain why the rank of $\mathbf{A}$ and the rank of $[\mathbf{A} \,|\, \mathbf{b}]$ can differ by at most 1. Here $\mathbf{b}$ is a column vector.

82. A set is linearly independent if and only if its indexed set is linearly independent. Explain why or find a counterexample.

TRUE–FALSE EXERCISES 1.3

1. If $\mathbf{A}$ is an $m \times n$ matrix and $n > m$, then the equation $\mathbf{Ax} = \mathbf{0}$ will have infinitely many solutions.

2. If the system of equations $\mathbf{Ax} = \mathbf{b}$ is inconsistent, then the augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ has a pivot position in the last column.

3. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{Ax} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$, then $m \geq n$.

4. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{Ax} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$, then $m < n$.

5. A system of equations $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution if and only if the columns of $\mathbf{A}$ form a linearly dependent set.

6. The vector $\mathbf{x} = [7, 6, -5]^T$ is in the kernel of the matrix $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \end{bmatrix}$

7. The kernel of $\begin{bmatrix} 3 & 7 & 6 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ consists solely of the vector $\mathbf{0}$.

8. If $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$, then $\mathbf{x} - \mathbf{y}$ is in the kernel of $\mathbf{A}$.

9. If $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$, then $\mathbf{y} + \mathbf{x}$ is in the kernel of $\mathbf{A}$.

10. If $\mathbf{Ax} = \mathbf{b} \neq \mathbf{0}$ and $\mathbf{Ay} = \mathbf{0}$, then $\mathbf{x} + \mathbf{y}$ is in the kernel of $\mathbf{A}$.

11. If $\mathbf{A}$ is row equivalent to $\mathbf{C}$ and if $\mathbf{B}$ is row equivalent to $\mathbf{C}$, then the kernels of $\mathbf{A}$ and $\mathbf{B}$ are the same.

12. If two matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, then their kernels are the same.

13. The kernel of a matrix $\mathbf{A}$ is the set of all vectors that can be expressed as linear combinations of the columns of $\mathbf{A}$.

14. Every solution of the equation $\mathbf{Ax} = \mathbf{b}$ is the sum of two vectors in the kernel of $\mathbf{A}$.

15. If the columns of a matrix form a linearly dependent set of vectors, then the kernel of that matrix contains nonzero vectors.

16. There exists a 4×4 matrix $\mathbf{A}$ having three pivot positions, namely, a_{11}, a_{23}, and a_{44}.

17. One of the pivot positions in this matrix is the $(1, 1)$-position (i.e., the position a_{11} in the usual notation): $\begin{bmatrix} 0 & 3 & 2 \\ 5 & 7 & 3 \\ 2 & -7 & 1 \end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 2 & 9 & 4 & 3 \\ 0 & 7 & 8 & 5 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 3 & -1 \end{bmatrix}$ is 4.

19. It is possible for a $p \times q$ matrix to have rank $p + 1$.

20. The rank of the matrix $\begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 3 & 4 & 8 \\ 4 & 2 & 7 & 5 \end{bmatrix}$ is 4.

21. Consider a system of equations $\mathbf{Ax} = \mathbf{b}$. If the ranks of the coefficient matrix and the augmented matrix are the same, then the system is consistent.

22. A system of equations $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution if and only if the rows of $\mathbf{A}$ form a linearly dependent set.

23. The rank of a matrix and the rank of its reduced row echelon form are the same.

24. If $\mathbf{A}$ is an $m \times n$ matrix and $m < n$, then the equation $\mathbf{Ax} = \mathbf{0}$ has some nontrivial solutions.

25. A system of linear equations is inconsistent if the reduced row echelon form of its augmented matrix has a pivot element in every column.

26. Consider a system of equations whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The *general solution* of this system can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

27. In the preceding exercise, the solution set can be described as the set of vectors $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$, where $t \in \mathbb{R}$.

28. Every homogeneous system of linear equations is consistent.

29. Every homogeneous system of linear equations has many solutions.

30. Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$. Every solution to the equation $\mathbf{Ax} = \mathbf{0}$ is a linear combination of the vectors $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$.

31. Let $\mathbf{u} = (1, 3, 2)$, $\mathbf{v} = (2, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

32. This set of three vectors is linearly independent: $\mathbf{u} = (1, 3, 5)$, $\mathbf{w} = (2, 0, 1)$, $\mathbf{z} = (0, 0, 3)$.

33. The set of columns in a matrix $\mathbf{A}$ is linearly independent if the equation $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution.

34. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix}$

Assume as known that $\mathbf{A}$ and $\mathbf{B}$ are row equivalent to each other. The set of columns in $\mathbf{A}$ is linearly independent.

35. Let $\mathbf{A} = \begin{bmatrix} 21 & 31 & 73 & 2 \\ 23 & 92 & -57 & 69 \\ 19 & 29 & 72 & 11 \end{bmatrix}$

The set of columns in this matrix is linearly independent.

36. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 29 \\ 7 & 0 & 13 \\ 13 & 0 & -53 \end{bmatrix}$

The set of columns in this matrix is linearly independent.

37. To determine whether a set of vectors is linearly dependent, we put the vectors into a matrix $\mathbf{A}$ as rows, and attempt to find nontrivial solutions to the corresponding homogeneous system of equations, $\mathbf{Ax} = \mathbf{0}$.

38. Let $\mathbf{A}$ be a 6×6 matrix such that the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each vector $\mathbf{b}$ in $\mathbb{R}^6$. A valid deduction from this information is that the equation $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions.

39. If each row of an $m \times n$ matrix $\mathbf{A}$ has a pivot position, then the rows of that matrix span $\mathbb{R}^n$.

40. If the columns of a matrix **A** span $\mathbb{R}^k$ and **A** has dimensions $k \times q$, then the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent for all vectors **b** in $\mathbb{R}^k$.

41. If the system of equations $\mathbf{Ax} = \mathbf{b}$ is inconsistent, then **b** is *not* in the span of the set of columns of **A**.

42. A system of linear equations $\mathbf{Ax} = \mathbf{b}$ is inconsistent if and only if **b** is in the span of the rows of **A**.

43. If the system of equations $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$, then the columns of **A** span $\mathbb{R}^m$.

44. This system has a nontrivial solution:

$$\begin{cases} 3x_1 - 5x_2 + x_3 = 0 \\ 2x_1 + 3x_2 + 9x_3 = 0 \end{cases}$$

45. Consider $x_1(\mathbf{C_3H_8}) + x_2(\mathbf{O_2}) = x_3(\mathbf{CO_2}) + x_4(\mathbf{H_2O})$. To balance this chemical equation, we can let $x_1 = 1, x_2 = 5, x_3 = 3$, and $x_4 = 4$.

46. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n vectors in $\mathbb{R}^m$, and suppose that $m > n$. A valid conclusion is that the set is linearly dependent.

47. For a 5×4 matrix **A**, the equation $\mathbf{Ax} = \mathbf{0}$ implies either $\mathbf{A} = \mathbf{0}$ or $\mathbf{x} = \mathbf{0}$.

48. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ and $\mathbf{x} = [1, -5, 3]^T$. Then **x** is in the kernel of **A**.

49. The kernel of $\begin{bmatrix} 4 & 8 & 7 \\ 3 & 4 & 0 \\ 7 & 0 & 0 \end{bmatrix}$ consists solely of the vector **0**.

50. If the columns of a matrix form a linearly independent set of vectors, then the kernel of that matrix contains only the zero vector.

51. The set of rows in the matrix

$$\begin{bmatrix} 1 & 0 & 29 \\ 7 & 0 & 13 \\ 13 & 0 & -53 \end{bmatrix}$$

is linearly independent.

52. If S is a linearly dependent set of vectors, then each vector in S is a linear combination of the other vectors in S.

53. If each column of an $m \times n$ matrix has a pivot position, then the columns of that matrix span $\mathbb{R}^n$.

54. Let **A** be an $m \times n$ matrix for which $n > m$. If the equation $\mathbf{Ax} = \mathbf{b}$ is consistent, then it has many solutions.

55. Let **A** be a $p \times q$ matrix, and suppose that $q > p$. Then it is possible in some cases for **A** to have rank $p + 1$.

56. The vector $\frac{3}{4}\mathbf{u}_2$ is in the span of the set of three vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

57. Consider the plane in $\mathbb{R}^3$ defined by the equation $x_1 - 4x_2 + 3x_3 = 0$. An alternative, parametric, description of this plane is $\mathbf{x} = t\mathbf{u} + s\mathbf{v}$, where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{u} = (4, 1, 0)$, and $\mathbf{v} = (-3, 0, 1)$.

58. If the matrix **A** has more rows than columns, then for some vectors **b** the system $\mathbf{Ax} = \mathbf{b}$ will be inconsistent. (Here **A** should be $m \times n$ and $\mathbf{b} \in \mathbb{R}^m$.)

59. If **x** and **y** are in the kernel of a matrix **A**, then $3\mathbf{x} - 2\mathbf{y}$ is also in the kernel.

60. The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in the kernel of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

61. The kernel of $\begin{bmatrix} 4 & 8 & 0 \\ 3 & 0 & 0 \\ 3 & 6 & 11 \end{bmatrix}$ consists solely of the vector $\mathbf{0}$.

62. If the rows of a matrix form a linearly independent set of vectors, then the kernel of that matrix contains only the zero vector.

63. One of the pivot positions in this matrix is the (1,2)-position (i.e., the position a_{12} in the usual notation): $\begin{bmatrix} 0 & 1 & 2 \\ 5 & 7 & 3 \\ 2 & -7 & 1 \end{bmatrix}$

64. The rank of $\begin{bmatrix} 5 & 3 & 6 & 11 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -15 & 9 \end{bmatrix}$ is 4.

65. If each row of an $m \times n$ matrix $\mathbf{A}$ has a pivot position, then for every $\mathbf{b}$ in $\mathbb{R}^m$ the system $\mathbf{Ax} = \mathbf{b}$ is consistent.

66. Let $\mathbf{A}$ be an $m \times n$ matrix for which $n > m$. Then, for every $\mathbf{b}$ in $\mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ has many solutions.

67. This set of three vectors is linearly independent:

$$\left\{ \mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

68. Consider a system of equations $\mathbf{Ax} = \mathbf{b}$. If the rank of the coefficient matrix is less than the rank of the augmented matrix, then the system is consistent.

69. The span of a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is the collection of all vectors that can be written in the form $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$, using integers $c_1, c_2, \ldots, c_k$.

70. If the columns of a matrix form a linearly independent set of vectors, then the kernel of that matrix contains only the zero vector.

71. Let $\mathbf{A}$ be an $m \times n$ matrix for which $n < m$. If the equation $\mathbf{Ax} = \mathbf{b}$ is consistent, then it has many solutions.

72. The rank of the matrix $\begin{bmatrix} 4 & 2 & 7 & 5 \\ 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ is 3.

73. This set of three vectors is linearly dependent:

$$\left\{ \mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

74. If $\text{Ker}(\mathbf{A}) = \text{Ker}(\mathbf{B})$, then $\mathbf{A} \sim \mathbf{B}$.

75. Every set of four vectors in $\mathbb{R}^3$ is linearly independent.

76. If $\mathbf{A}$ is a $p \times q$ matrix and if $q > p$, then every equation of the form $\mathbf{Ax} = \mathbf{b}$ (where $\mathbf{b} \in \mathbb{R}^p$) will have a free variable and therefore will have infinitely many solutions.

77. If $\mathbf{A}$ is an $m \times n$ matrix and $n < m$, then the equation $\mathbf{Ax} = \mathbf{0}$ will have infinitely many solutions.

78. Consider a system of equations $\mathbf{Ax} = \mathbf{b}$. If the rank of the coefficient matrix is the same as the rank of the augmented matrix, then the system is consistent.

79. If $\mathbf{x}$ and $\mathbf{y}$ are in the kernel of a matrix $\mathbf{A}$, then $\mathbf{x} - \mathbf{y}$ is also in the kernel.

80. If the system of equations $\mathbf{Ax} = \mathbf{b}$ has one or more free variables, then the system has many solutions.

81. If the system $\mathbf{Ax} = \mathbf{0}$ has a free variable, then the system has many solutions. In this case, the kernel (or *null space*) of $\mathbf{A}$ has dimension 1 or greater.

82. The zero element in the matrix $\begin{bmatrix} 0 & 2 & 5 \\ 3 & 1 & 1 \\ 4 & 3 & 6 \end{bmatrix}$ does not occupy a pivot position.

MULTIPLE-CHOICE EXERCISES 1.3

Always select the first correct answer.

1. Consider $\left[\begin{array}{ccccc|c} 3 & -7 & 11 & 9 & 0 & 5 \\ 0 & 6 & -5 & 4 & 1 & 17 \\ 0 & 0 & 0 & 0 & 9 & 0 \end{array}\right]$

What are all the free variables in solving a system with this augmented matrix?

a. x_1, x_2, x_5 **b.** x_3 and x_4 **c.** x_4
d. There are *no* free variables.
e. None of these.

2. What are all the free variables in the system of equations whose coefficient matrix is

$$\begin{bmatrix} 0 & 7 & 9 & 4 & 3 & 0 & 2 \\ 0 & 0 & 0 & 6 & 5 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 \end{bmatrix}?$$

a. x_3, x_4, x_5 **b.** x_1, x_3, x_5
c. x_1, x_3, x_5, x_7 **d.** x_2, x_4, x_6
e. None of these.

3. Consider $\left[\begin{array}{cccccc|c} 0 & 4 & 3 & 4 & 6 & 11 & 5 \\ 0 & 0 & 0 & 7 & 2 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 \end{array}\right]$

A system of linear equations has the *augmented* matrix shown. What are all the free variables?

a. x_2, x_4, x_6 **b.** x_3, x_5 **c.** x_1, x_3, x_5
d. x_1, x_3, x_5, x_7 **e.** None of these.

4. Consider $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \end{array}\right]$

The general solution of the system having this augmented matrix is

a. $(-1, 1, 0, -4, 0)$
b. $x_1 = -1 - x_5$, $x_2 = 1 + 3x_5$, x_3 free, $x_4 = -4 - 5x_5$, x_5 free
c. $x_1 = -3 - 7x_5 + 2x_2$, x_2 free, x_3 free, $x_4 = -4 - 5x_5$, x_5 free
d. $x_1 + x_5 = -1$, $x_2 - 3x_5 = 1$, $x_4 + 5x_5 = -4$, x_3 free
e. None of these.

5. Consider $\begin{cases} x_3 - x_1 = 14 \\ x_2 - x_1 = 3 \\ x_3 - 3x_2 + x_1 = 7 \end{cases}$

A solution, (x_1, x_2, x_3), of this system is
a. $(-26, -11, 32)$ **b.** $(-2, 1, 12)$
c. $(1, 0, 15)$ **d.** $(1, 15, 4)$
e. None of these.

6. Consider $\begin{cases} x_2 + 5x_3 = -4 \\ x_1 + 4x_2 + 3x_3 = -2 \\ 2x_1 + 7x_2 + 2x_3 = -1 \end{cases}$

A solution vector for the system shown is
a. $(0, 4, 1)$ **b.** $(-3, 1, -1)$
c. $(4, 1, -1)$ **d.** $(1, 1, -2)$
e. None of these.

7. Which set is linearly independent?

a. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \right\}$

d. The set $\{p_1, p_2, p_3\}$ where $p_1(t) = t + t^2$, $p_2(t) = t^2 + t^3$, $p_3(t) = t + t^3$

e. None of these.

8. Let K be the kernel of a matrix $\mathbf{A}$; that is, $K = \{\mathbf{x} : \mathbf{Ax} = \mathbf{0}\}$. Which assertion may be false?

a. If $\mathbf{x}$ and $\mathbf{y}$ are in K, then so is $\mathbf{x} - \mathbf{y}$.

b. If $\mathbf{Ay} = \mathbf{b}$ and $\mathbf{Ax} = 2\mathbf{b}$, then $2\mathbf{y} - \mathbf{x}$ is in K.

c. If $\mathbf{Ay} = \mathbf{b}$ and $\mathbf{x}$ is in K, then $\mathbf{A}(\mathbf{x}+2\mathbf{y}) = \mathbf{b}$.

d. If $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$, then $\mathbf{y}$ is the sum of $\mathbf{x}$ and a member of K.

e. None of these.

9. For which matrix $\mathbf{A}$ does the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ have nontrivial solutions?

a. $\begin{bmatrix} 5 & 3 & 1 & 4 \\ -1 & 6 & 3 & 5 \\ 2 & -4 & 1 & 9 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & -1 & 3 & 5 \\ 1 & 0 & 6 & 9 \\ 5 & 7 & 7 & 8 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 9 & 8 & 2 \\ 4 & 4 & 7 & 1 \\ 3 & 2 & 1 & 5 \\ 5 & 5 & 4 & 4 \end{bmatrix}$ **d.** $\begin{bmatrix} 6 & 4 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 2 \\ 5 & 0 & 3 \end{bmatrix}$

e. None of these.

10. Consider $\begin{bmatrix} 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 6 & -5 & 4 & 1 & 17 \\ 3 & -7 & 11 & 9 & 0 & 5 \end{bmatrix}$

What are all the free variables in solving a system with this augmented matrix?

a. x_1, x_2, x_5, **b.** x_3, x_4 **c.** x_4

d. There are *no* free variables

e. None of these.

11. For what value of c is the set of vectors $\{[1, 2, 3, 1], [1, 3, 3, 2], [1, 5, 6, 7], [1, 1, 5, c]\}$ linearly dependent?

a. 0 **b.** 2 **c.** 5 **d.** 7

e. None of these.

12. Let $\mathbf{u} = (1, 4, 7)$, $\mathbf{v} = (2, 8, -11)$, $\mathbf{w} = (3, 12, -4)$, $\mathbf{z} = (0, 0, 0)$. Which set is linearly independent?

a. $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, **b.** $\{\mathbf{v}, \mathbf{w}\}$

c. $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ **d.** $\{\mathbf{v}, \mathbf{w}, \mathbf{z}\}$

e. None of these.

13. Let $\mathbf{v}_1 = (1, 3, -3)$, $\mathbf{v}_2 = (3, 10, -1)$, $\mathbf{v}_3 = (-2, -1, h)$. For what value of h is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent?

a. 0 **b.** 6

c. 64 **d.** 46

e. None of these.

14. Let $\mathbf{v} = (0, 0, 0)$, $\mathbf{w} = (1, 2, 3)$, $\mathbf{y} = (2, 4, -6)$, $\mathbf{z} = (3, 6, -3)$, $\mathbf{u} = (1, 0, 1)$. Which set is linearly independent?

a. $\{\mathbf{w}, \mathbf{y}\}$ **b.** $\{\mathbf{w}, \mathbf{y}, \mathbf{z}\}$ **c.** $\{\mathbf{w}, \mathbf{y}, \mathbf{v}\}$

d. $\{\mathbf{z}, \mathbf{v}\}$ **e.** None of these.

15. (Continuation.) Use the vectors $\mathbf{v}, \mathbf{w}, \mathbf{y}, \mathbf{z}, \mathbf{u}$ from the preceding problem. Which set is linearly dependent?

a. $\{\mathbf{w}, \mathbf{y}\}$ **b.** $\{\mathbf{u}, \mathbf{z}\}$ **c.** $\{\mathbf{u}, \mathbf{y}\}$

d. $\{\mathbf{u}, \mathbf{z}, \mathbf{y}, \mathbf{w}\}$ **e.** None of these.

16. Which set is linearly independent?

a. $\{(1, 3, 2), (-2, 1, 4), (3, 3, 4), (1, 0, 2)\}$

b. $\{(1, 3, 2), (-2, 1, 4), (19, 15, -10)\}$

c. $\{p_1, p_2, p_3\}$, where $p_1(x) = x^2 - 3x + 7$, $p_2(x) = 2x^2 + x - 3$, and $p_3(x) = x^2 + 11x - 27$

d. $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$

e. None of these.

17. If $\mathbf{A}$ is a 5×3 matrix, which conclusion is valid?

a. The set of rows is linearly dependent.

b. The set of columns is linearly independent.

c. The set of rows is linearly independent.

d. The set of columns is linearly dependent.

e. None of these.

18. If **A** is a 3×5 matrix, which conclusion is valid?

a. The set of rows is linearly dependent.
b. The set of columns is linearly independent.
c. The set of rows is linearly independent.
d. The set of columns is linearly dependent.
e. None of these.

19. Let **A** be a 3×4 matrix whose columns form a linearly independent set. Which conclusion is justified?

a. The set of rows in **A** is linearly dependent.
b. The equation $\mathbf{Ax} = \mathbf{b}$ is consistent for every **b** in $\mathbb{R}^4$.
c. The equation $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution.
d. There is a matrix **AB** such that $\mathbf{AB} = \mathbf{I}_4$.
e. None of these.

20. Let **A** be a 4×3 matrix whose columns form a linearly independent set. Which conclusion is justified?

a. The set of rows in **A** is linearly dependent.
b. The equation $\mathbf{Ax} = \mathbf{b}$ is consistent for every **b** in $\mathbb{R}^4$.
c. The equation $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution.
d. There is a matrix **B** such that $\mathbf{AB} = \mathbf{I}_4$.
e. None of these.

21. Which condition implies that the system of equations $\mathbf{Ax} = \mathbf{b}$ has a solution?

a. The columns of **A** form a linearly independent set.
b. The vector **b** is in the span of the set of columns of **A**.
c. The equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of **A** has no zero row.
e. None of these.

22. Which point is *not* in the kernel of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 4 \end{bmatrix}$?

a. $\begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ **c.** $\begin{bmatrix} -16 \\ 8 \\ -2 \end{bmatrix}$

d. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **e.** None of these.

23. Let $\mathbf{A} = [1 \quad -2 \quad -5]$. (This is a 1×3 matrix.) Every solution of $\mathbf{Ax} = \mathbf{0}$ is a linear combination of which pair of vectors?

a. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

e. None of these.

24. Let **A** be an $m \times n$ matrix whose rows span $\mathbb{R}^n$. What conclusion is valid?

a. $m \geq n$
b. Each row of **A** has a pivot element.
c. The set of rows in **A** is linearly independent.
d. The set of columns in **A** is linearly dependent.
e. None of these.

25. Let **A** be an $m \times n$ matrix of rank n. Which conclusion is justified?

a. The kernel of **A** is $\{\mathbf{0}\}$.
b. The range of **A** is $\mathbb{R}^m$.
c. $m \leq n$
d. $n > m$
e. None of these.

26. Let **A** be an $m \times n$ matrix whose columns span $\mathbb{R}^m$. Which one of these conclusions is unjustified?

a. $n \geq m$
b. The rows of **A** span $\mathbb{R}^n$
c. For every **b** in $\mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ is consistent.
d. Every row of **A** has a pivot position.
e. None of these.

27. Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are specified in a vector space. Four sets of vectors are defined: $S = \{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$, $T = \{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}+3\mathbf{v}\}$, $Q = \{\mathbf{u}, 2\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{w}\}$, $Z = \{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{w}+\mathbf{u}, \mathbf{w}\}$. Select the largest set that is necessarily linearly dependent.

a. S, T **b.** S, T, Q, Z **c.** S, T, Z
d. T, Q **e.** None of these.

28. We form sets $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, 0\}$, $T = \{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_1\}$, $Q = \{\mathbf{v}_1, 2\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3\}$, $W = \{\mathbf{v}_1, \mathbf{v}_2 + \mathbf{v}_1, \mathbf{v}_3 + \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3\}$ from three different vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^7$. Give a list of *all* the sets that are linearly dependent.

a. S, T, Q, W **b.** S, T, Q **c.** S, T, W
d. T, Q **e.** None of these.

29. Let $\mathbf{u} = (1, 4, 7)$, $\mathbf{v} = (2, 8, -11)$, $\mathbf{w} = (3, 12, -4)$, and $\mathbf{z} = (0, 0, 0)$. Which set is linearly independent?

a. $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ **b.** $\{\mathbf{v}, \mathbf{w}\}$ **c.** $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$
d. $\{\mathbf{v}, \mathbf{w}, \mathbf{z}\}$ **e.** None of these.

30. Find a vector **x** having integer components x_i so that this chemical equation is balanced: $x_1(\mathbf{Na_3PO_4}) + x_2(\mathbf{Ba(NO_3)_2}) = x_3(\mathbf{Ba_3(PO_4)_2}) + x_4(\mathbf{NaNO_3})$ involving sodium phosphate, barium nitrate, barium phosphate, and sodium nitrate. The elements are sodium **Na**, phosphorus **P**, oxygen **O**, barium **Ba**, and nitrogen **N**.

a. $\mathbf{x} = (1, 2, 3, 4)$ **b.** $\mathbf{x} = (3, 2, 6, 4)$
c. $\mathbf{x} = (2, 3, 1, 6)$ **d.** $\mathbf{x} = (3, 5, 2, 1)$
e. None of these.

31. Find integers x, y, z, w so that this chemical equation is balanced: $x(\mathbf{B_2S_3}) + y(\mathbf{H_2O}) = z(\mathbf{H_3BO_3}) + w(\mathbf{H_2S})$ involving boron **B**, sulfur **S**, hydrogen **H**, oxygen **O**, and water $\mathbf{H_2O}$.

a. $x = 6, y = 2, z = 4, w = 8$
b. $x = 1, y = 6, z = 2, w = 3$
c. $x = 1, y = 2, z = 3, w = 4$
d. $x = 6, y = 1, z = 2, w = 3$
e. None of these.

32. Find a vector **x** with integer components x_i so that this chemical equation is balanced: $x_1(\mathbf{PbN_6}) + x_2(\mathbf{Mn_2O_8}) = x_3(\mathbf{Pb_3O_4}) + x_4(\mathbf{MnO_2}) + x_5(\mathbf{NO})$ involving lead **Pb**, nitrogen **N**, manganese **Mn**, and oxygen **O**.

a. $\mathbf{x} = (15, 44, 5, 22, 88)$
b. $\mathbf{x} = (5, 4, 1, 2, 8)$
c. $\mathbf{x} = (6, 11, 2, 22, 36)$
d. $\mathbf{x} = (2, 3, 1, 4)$
e. None of these.

33. Let $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (3, 1, 3)$, and $\mathbf{v}_3 = (5, 3, 3)$. This set of three vectors is linearly dependent because $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$. The coefficients a, b, and c are

a. $a = 6, b = 2, c = 4$
b. $a = 1, b = 6, c = 2$
c. $a = 2, b = 1, c = -1$
d. $a = 6, b = 1, c = 2$
e. None of these.

34. All solutions of $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & 4 & -3 & 0 \end{array}\right]$ are multiples of

a. $(5, -2, 1)$
b. $(5, -3, 1)$
c. $(0, 0, 0)$
d. $(6, 1, 2)$
e. None of these.

35. The general solution of $5x_1 + 2x_2 - x_3 = 0$ is the span of this pair:

a. $(1, 2, 1), (3, 1, 2)$
b. $(-\frac{2}{5}, 1, 0), (\frac{1}{5}, 0, 1)$
c. $(-2, 5, 0), (1, 0, 5)$
d. $(10, 0, 50), (-20, 50, 0)$
e. None of these.

36. Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 2 & 5 & 9 & 0 \\ 3 & 6 & 12 & 0 \end{array}\right]$$

The general solution of this system is all multiples of

a. $(-2, -1, -1)$ **b.** $(-2, -1, 1)$
c. $(2, 1, -1)$ **d.** $(-1, \frac{1}{2}, -\frac{1}{2})$
e. None of these.

37. What is the general solution of a system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 3 \\ 2 & 5 & 9 & 7 \\ 3 & 6 & 12 & 11 \end{array}\right]?$$

a. $(1, \frac{1}{2}, \frac{1}{2})$
b. $(13/3, -\frac{1}{3}, 0) + t(-2, -1, 1)$
c. $(13, -1, 0) + s(2, 1, -1)$
d. $(10, 0, 50) + x_3(1, \frac{1}{2}, -\frac{1}{2})$
e. None of these.

38. Which matrix does not have rank 2?

a. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -2 & -2 \\ 3 & 1 & 3 \end{bmatrix}$

e. None of these.

39. Students Octavio and Valeria compute the kernel of the matrix $\begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 5 & 2 & 1 \end{bmatrix}$. Octavio claims that the kernel is the span of this pair of vectors $(7, -3, 0, 1)$ and $(4, -2, 1, 0)$. But Valeria insists that the kernel is the span of $(-20, 10, -5, 0)$ and $(21, -9, 0, 3)$. Which is right?

a. Only Valeria **b.** Only Octavio
c. Both of them. **d.** Neither of them.

40. Let $\mathbf{A}$ be a 5×4 matrix whose columns form a linearly independent set. Which conclusion is justified?

a. The equation $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^5$.
b. The set of rows in $\mathbf{A}$ is linearly dependent.
c. There is a matrix $\mathbf{AB}$ such that $\mathbf{AB} = \mathbf{I}_5$.
d. The equation $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution.
e. None of these.

41. Let K be the kernel of a matrix $\mathbf{A}$; that is, $K = \{\mathbf{x} : \mathbf{Ax} = \mathbf{0}\}$. Which assertion must be true?

a. If $\mathbf{x}$ and $\mathbf{y}$ are in K, then so is $\mathbf{x} - \mathbf{y}$.
b. If $\mathbf{Ay} = \mathbf{b}$ and $\mathbf{Ax} = 2\mathbf{b}$, then $2\mathbf{y} - \mathbf{x}$ is in K.
c. If $\mathbf{Ay} = \mathbf{b}$ and $\mathbf{x}$ is in K, then $\mathbf{A}(\mathbf{x} + 2\mathbf{y}) = \mathbf{b}$.
d. If $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$, then $\mathbf{y}$ is the sum of $\mathbf{x}$ and a member of K.
e. None of these.

42. For which matrix $\mathbf{A}$ does the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ have nontrivial solutions?

a. $\begin{bmatrix} 5 & 3 & 1 & 4 \\ -1 & 6 & 3 & 5 \\ 2 & -4 & 1 & 9 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & -1 & 3 & 5 \\ 1 & 0 & 6 & 9 \\ 5 & 7 & 7 & 8 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 9 & 8 & 2 \\ 4 & 4 & 7 & 1 \\ 3 & 2 & 1 & 5 \\ 5 & 5 & 4 & 4 \end{bmatrix}$ **d.** $\begin{bmatrix} 6 & 4 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 2 \\ 5 & 0 & 3 \end{bmatrix}$

e. None of these.

43. Define three polynomials:
$u_1(t) = 7t^5 - 4t^2 + 3$
$u_2(t) = 2t^5 + 5t^2$
$u_3(t) = 8t^5 - 23t^2 + 6$
Which of the following equations establishes the linear dependence of the set $\{u_1, u_2, u_3\}$?
a. $2u_1(t) - 7u_2(t) = 6 - 43t^2$
b. $4u_2(t) - u_3(t) = 6 - 43t^2$
c. $2u_1(t) - 3u_2(t) = u_3(t)$
d. $tu_1(t) + tu_2(t) = 9t^6 + t^3 + 3t$
e. None of these.

44. Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 3 & 4 & -2 & 5 \\ 2 & -1 & 0 & 1 \end{bmatrix}$

The rank of **A** is
a. 2 **b.** 1 **c.** 4 **d.** 3
e. None of these.

45. Let **A** be a 5×4 matrix whose columns form a linearly independent set. Which conclusion is justified?
a. The equation $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^5$.
b. The set of rows in **A** is linearly dependent.
c. There is a matrix **AB** such that $\mathbf{AB} = \mathbf{I}_5$.
d. The equation $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution.
e. None of these.

COMPUTER EXERCISES 1.3

1. Consider the following scenario. An unidentified object is observed in the night sky during a period of several days, and accurate coordinates of this object have been made available by astronomers. The orbit of such an object should be a *conic section*: circle, ellipse, hyperbola, parabola, or a straight line. This means that if the locations are plotted on a plane, one of these types of conic sections should be evident. In particular, if the orbit is an ellipse, the object will return after a certain number of years, whereas if the orbit is a parabola or hyperbola it will not return. An arbitrary conic section in (x, y) coordinates should have the form $ax^2 + bxy + cy^2 + dx + ey = 1$.

a. Find the values of a, b, c, d, and e from these five points on the orbit: $(1.8, 3.1)$, $(1.4, 1.9)$, $(2.5, 1.2)$, $(4.0, 1.6)$, $(4.8, 2.5)$.

b. Determine whether the orbit is elliptical, parabolic, or hyperbolic.

c. Find a formula by which the y-values can be computed from the x-values in this orbit. Remember, there may be two y-values for a given x, since there is a quadratic equation to be solved.

d. If you have suitable facilities at your disposal, obtain a plot of the orbit.